M2S1 - ASSESSED COURSEWORK 3

To be handed in no later than Tuesday, 30th November

1.(a) Suppose that X and Y are continuous random variables with joint pdf $f_{X,Y}$. Let random variable U be defined by $U = XY$. By considering the joint transformation $(X,Y) \longrightarrow (U,V)$ where $V = X$, show that

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,Y}(x, u/x)|x|^{-1}\, dx$$

Suppose that

$$f_{X,Y}(x,y) = c(\alpha,\beta)\ x^{\alpha-1}(1-x)^{\beta-1}\ \ c(\alpha+\beta,\gamma)\ y^{\alpha+\beta-1}(1-y)^{\gamma-1} \qquad 0 < x < 1,\ 0 < y < 1.$$

and zero otherwise, for $\alpha, \beta, \gamma > 0$, where normalizing constants $c(\alpha,\beta)$ and $c(\alpha+\beta,\gamma)$ are defined by

$$\frac{1}{c(a,b)} = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$$

Find the marginal pdf of random variable $U = XY$.

[*Hint: as X and Y are random variables with ranges $\mathbb{X} = \mathbb{Y} = (0,1)$, we have $XY < X$.*]

(b) Suppose now that X and Y have joint pdf given by

$$f_{X,Y}(x,y) = c\ xy(1-x-y) \qquad 0 < x < 1,\ 0 < y < 1,\ 0 < x+y < 1.$$

for some constant $c > 0$. Find the covariance of X and Y.

2. The continuous random variable X with range $\mathbb{X} = \mathbb{R}$ has pdf given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} \qquad x \in \mathbb{R}$$

Suppose that $X_1, ..., X_k$ are independent random variables having the same distribution as X.

(a)(i) Find the mgf of random variable X.

(ii) Find the pdf and the mgf of random variable Y defined by $Y = \sigma X + \mu$ for parameters μ and σ.

(iii) Find the pdf of random variable S defined by

$$S = \sum_{i=1}^{k} X_i$$

(b) Let random variable T be defined by $T = X^2$.

(i) Find the pdf of T.

(ii) Find the mgf of T.

(iii) Find the mgf of random variable U defined by

$$U = \sum_{i=1}^{k} X_i^2.$$

SUPPLEMENTARY QUESTIONS

(not to be handed in)

1. Suppose that X_1 and X_2 are independent continuous random variables with pdf f_X and cdf F_X. Let Y_1 and Y_2 be the smaller and larger of the pair $\{X_1, X_2\}$ respectively. If constant c is defined by $F_X(c) = p$, for some p where $0 \leq p \leq 1$, evaluate $\mathrm{P}[Y_2 > c]$.

2. Suppose that $X_1, ..., X_k$ are i.i.d continuous random variables with density function

$$f_X(x) \;=\; \lambda e^{-\lambda x} \quad x > 0$$

and zero otherwise, for some $\lambda > 0$. Find the marginal pdf of the minimum order statistic, Y_1.

3. Suppose that $X_1, ..., X_k$ are i.i.d continuous random variables with density function

$$f_X(x) \;=\; 1/x^2 \qquad 1 \leq x < \infty$$

Write down the joint pdf of the order statistics $Y_1, ..., Y_k$ derived from $X_1, ..., X_k$, and the marginal pdf of the smallest order statistic Y_1, and the marginal pdf of the largest order statistic Y_k. For k odd, $k = 2n + 1$ say, derive the marginal pdf of the order statistic Y_n.

4. Suppose that $X_1, ...X_k$ are i.i.d. continuous random variables with pdf that is constant on $(0, 1)$, zero otherwise, and let Y_k be the corresponding largest order statistic. Find the smallest k such that $\mathrm{P}[Y_k \geq 0.99] \geq 0.95$. Find also the pdf of the range random variable, $R = Y_k - Y_1$.

5. Suppose that X is a continuous random variable with pdf given by

$$f_X(x) = 5x^4 \qquad 0 \leq x \leq 1$$

and zero otherwise. Suppose that f_X above is the pdf for the marks (out of 100, and rescaled to 0-1) for a given student in a test. If test results for k tests are regarded as i.i.d. random variables having pdf f_X, let Y_1 be the random variable corresponding to the minimum score obtained in the k tests, evaluate $\mathrm{P}[Y_1 \leq 0.75]$ if $k = 3, 4, 5$. Find the pdf of Y_1, and the limiting probability distribution of Y_1 as the number of tests, k becomes large.

6. Suppose that random variable X has mgf given by $M_X(t) = 1/8e^t + 1/4e^{2t} + 5/8e^{3t}$. Find the probability distribution, and the expectation and variance of X.

7. Suppose that random variable X has mgf given by $M_X(t) = \left(1 - \theta + \theta e^t\right)^n$, for some θ, where $0 \leq \theta \leq 1$. Obtain a power series expansion for $M_X(t)$, and hence identify $\mathrm{E}_{f_X}[X^r]$ for $r = 1, 2, 3, ...$

8. Suppose that X is a continuous random variable with pdf

$$f_X(x) \;=\; \exp\{-(x + 2)\} \qquad -2 < x < \infty$$

Find the mgf of X, and hence find the expectation and variance of X.

9. Suppose that X is a random variable with mass function/pdf f_X and mgf M_X which exists for t in some neighbourhood of zero, $-h < t < h$, $h > 0$. Recall that the cumulant generating function of X, K_X, is defined by $K_X(t) = \log[M_X(t)]$. Prove that

$$\frac{d}{dt}\{K_X(t)\}_{t=0} = \mathrm{E}_{f_X}[X] \qquad\qquad \frac{d^2}{dt^2}\{K_X(t)\}_{t=0} = \mathrm{Var}_{f_X}[X]$$