M2S1 - ASSESSED COURSEWORK 1

To be handed in no later than Friday, 29th October

1. (a) A biased coin is tossed repeatedly, with tosses independent; the probability of the coin showing Heads on any toss is p. Let p_n denote the probability that an even number of Heads have been obtained after n tosses, and define $p_0 = 1$ (assume zero is an even number).

Show that, for $n \geq 1$,

$$p_n = (1 - 2p)p_{n-1} + p.$$

(b) Find a solution to this difference equation, valid for all $n \geq 0$, of the form

$$p_n = A + B\lambda^n$$

where A, B and λ are constants.

Prove that if $p < 1/2$ then $p_n > 1/2$ for all $n \geq 1$, and find the limiting value of p_n as $n \longrightarrow \infty$.

2. (a) Suppose that N fair, standard dice are rolled, where the scores on the N dice are independent, where N is unknown. Let A_i be the event that $N = i$, for $i \geq 1$, and suppose that

$$\mathrm{P}(A_i) = \frac{1}{2^i} \quad i = 1, 2, \ldots .$$

Let S denote the sum of the scores on the N dice.

Find the probability that

(i) $S = 4$, given that N is even. (ii) $N = 2$, given that $S = 4$.

(b) The *genotype* (genetic composition) of any individual at a pair of *loci* (sites) is defined by two pairs of *alleles*. Suppose that, at each locus (site), each allele in the pair is either 0 or 1, selected independently with probability $1 - p$ and p respectively, so that the probabilities of each possible genotype at a single locus are given by

Genotype	00	01	11
Probability	$(1-p)^2$	$2p(1-p)$	p^2

For example, the genotype of an individual at a pair of loci may be $(00, 01)$, $(11, 11)$ etc.

Suppose that, attached to each allele is a fixed score, specifically, a score of -5 is attached to allele 0, and a score of 5 is attached to allele 1, so that, for example

$$(00, 01) \quad \text{has a score of} \quad (-5) + (-5) + (-5) + 5 = -10.$$

(i) List the possible scores that may be obtained in this two locus, two allele system, and find the probability that each of these scores is obtained.

(ii) Find the probability that an individual is *doubly homozygous* (that is, has genotype $(00, 00)$ or $(11, 11)$) given that their total score is greater than zero.

SUPPLEMENTARY QUESTIONS

(not to be handed in)

S1. Two players contest a match consisting of a series of games, in which the match is won by the player who first wins three games, unless the score becomes tied at two games each in which case the match continues until a player gains a lead of two games over the opponent.

Suppose that the probability that player 1 wins a game is p, that games are probabilistically independent, and that no draws are possible.

Let A be the event that player 1 wins the match. Show that

$$\mathrm{P}(A) \;=\; \frac{p^3\left(4-5p+2p^2\right)}{(1-2p+2p^2)} \;=\; g(p), \text{ say.}$$

Evaluate $g(p)+g(1-p)$ and comment on this result.

S2. A simple model for weather forecasting involves classifying days as either Fine or Wet, and then assuming that the weather on a given day will be the same as the weather on the preceding day with probability p.

Suppose that the probability of fine weather on day indexed 1 (say Jan 1st) is denoted θ. Let θ_n denote the probability that day indexed n is Fine.

Prove the recursion formula

$$\theta_n \;=\; p\theta_{n-1}+(1-p)(1-\theta_{n-1}),\;\; n=2,3,...$$

for θ_n, where $\theta_1 \equiv \theta$.

Using the recursion formula, obtain an expression for θ_n explicitly, and find the limiting value of θ_n as $n \longrightarrow \infty$.

S3. Consider two coins, of which one is normal and the other has a Head on both sides. One of the coins is selected with equal probability $1/2$, and tossed repeatedly, with tosses considered independent.

Evaluate the (posterior) conditional probability that the selected coin is normal, given that the first n tosses are Heads.

Now consider two coins, of which one is normal and the other is biased so that the probability of obtaining a Head on a single toss is $p > 1/2$. Again, one of the coins is selected with equal probability, and tossed n times independently. Let E be the event that the n tosses result in k Heads and $n-k$ Tails, and let F be the event that the coin is fair. Find expressions for

$$\text{(i) } \mathrm{P}(E) \qquad \text{(ii) } \mathrm{P}(F|E)$$

S4. The probability that a tree has n flowers is given by $(1-p)p^n$ for $n = 0, 1, 2, ...$ Each flower has probability $2/3$ of being pollinated and producing fruit, and each fruit has probability of $1/4$ of not ripening fully. It can be assumed that each developmental stage is independent of the others.

Deduce that the probability of a flower producing a ripe fruit is $1/2$. Given that a tree bears r ripe fruit, calculate the conditional probability that it originally had n flowers.

S5. A company is to introduce mandatory drug testing for its employees. The test used is very accurate, in that it it gives a correct positive test (detects drugs when they are present in a blood sample) with probability 0.99, and a correct negative test (does not detect drugs when they are not present) with probability 0.98. If an individual tests positive on the first test, a second blood sample is tested. It is assumed that only 1 in 5000 employees actually does provide a blood sample with drugs present.

What is the probability that the presence of drugs in a blood sample is detected correctly, given

(i) a positive result on the first test only (ii) a positive result on both first and second tests