

M2S1 - EXERCISES 8: SOLUTIONS

1. As $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$, and given that $T_1 = \bar{X}$, then using elementary properties of expectations, we have

$$E_{f_{T_1}} [T_1] = \frac{1}{n} \sum_{i=1}^n E_{f_{X_i}} [X_i] = \frac{1}{n} \sum_{i=1}^n \lambda = \lambda$$

so that T_1 is an *unbiased* estimator of λ . Furthermore

$$\begin{aligned} T_2 &= \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1} \sum_{i=1}^n X_i^2 - \frac{1}{(n-1)} \sum_{i=1}^n (\bar{X})^2 = \frac{1}{n-1} \sum_{i=1}^n X_i^2 - \frac{1}{(n-1)} \sum_{i=1}^n \left(\frac{1}{n} \sum_{i=1}^n X_i \right)^2 \\ &= \frac{1}{n-1} \sum_{i=1}^n X_i^2 - \frac{1}{n^2(n-1)} \sum_{i=1}^n \left(\sum_{i=1}^n X_i \right)^2 = \frac{1}{n-1} \sum_{i=1}^n X_i^2 - \frac{1}{n(n-1)} \left(\sum_{i=1}^n X_i \right)^2 \end{aligned}$$

From properties of expectations, variances, and the Poisson distribution,

$$E_{f_X} [X^2] = \text{Var}_{f_X} [X] + \{E_{f_X} [X]\}^2 = \lambda + \lambda^2 = \lambda(\lambda + 1)$$

Now, from properties of independent Poisson random variables $Y_n = \sum_{i=1}^n X_i \sim \text{Poisson}(n\lambda)$ so therefore, taking expectations in the above

$$\begin{aligned} E_{f_{T_2}} [T_2] &= \frac{1}{n-1} \sum_{i=1}^n E_{f_{X_i}} [X_i^2] - \frac{1}{n(n-1)} E_{f_{Y_n}} [Y_n^2] = \frac{1}{n-1} \sum_{i=1}^n \lambda(\lambda + 1) - \frac{1}{n(n-1)} n\lambda(n\lambda + 1) \\ &= \frac{n}{n-1} \lambda(\lambda + 1) - \frac{1}{n-1} \lambda(n\lambda + 1) = \frac{1}{n-1} [n\lambda(\lambda + 1) - \lambda(n\lambda + 1)] = \lambda \end{aligned}$$

2. From the theorem in lectures, and by properties of the Gamma distribution, we can write

$$V_n = \frac{(n-1)s_n^2}{\sigma^2} \sim \chi_{n-1}^2 \equiv \text{Gamma} \left(\frac{n-1}{2}, \frac{1}{2} \right) \implies V_n = \sum_{i=1}^n X_i$$

where $X_i \sim \chi_1^2 \equiv \text{Gamma} \left(\frac{1}{2}, \frac{1}{2} \right)$, so that, using the Gamma expectation and variance results

$$E_{f_{X_i}} [X_i] = \frac{1/2}{1/2} = 1 = \mu \quad \text{Var}_{f_{X_i}} [X_i] = \frac{1/2}{(1/2)^2} = 2 = \sigma^2$$

say. Hence, by the Central Limit Theorem,

$$\frac{V_n - (n-1)\mu}{\sqrt{(n-1)\sigma^2}} = \frac{V_n - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} Z \sim N(0, 1)$$

Hence, substituting in the definition for V_n ,

$$\frac{\frac{(n-1)s_n^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} = \frac{\sqrt{n-1}(s_n^2 - \sigma^2)}{\sigma^2 \sqrt{2}} \xrightarrow{d} Z \sim N(0, 1)$$

and finally, by a location/scale transformation to $Z_n = \sigma^2 + \frac{\sigma^2 \sqrt{2}}{\sqrt{n-1}} Z$, we have

$$s_n^2 \xrightarrow{d} Z_n \sim N \left(\sigma^2, \frac{2\sigma^4}{n-1} \right)$$

3. $X_1, \dots, X_n \sim \text{Gamma}(\alpha, \beta)$ so that $E_{f_{X_i}}[X_i] = \alpha/\beta$ and $\text{Var}_{f_{X_i}}[X_i] = \alpha/\beta^2$ so that

$$E_{f_{X_i}}[X_i^2] = \text{Var}_{f_{X_i}}[X_i] + \left\{E_{f_{X_i}}[X_i]\right\}^2 = \frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha(\alpha+1)}{\beta^2}$$

Hence for the method of moments estimators $\hat{\alpha}_{MM}$ and $\hat{\beta}_{MM}$, need to solve the following:

FIRST MOMENT	Solve	$\frac{1}{n} \sum_{i=1}^n x_i = \bar{x} = \frac{\alpha}{\beta}$
SECOND MOMENT	Solve	$\frac{1}{n} \sum_{i=1}^n x_i^2 = (\bar{x})^2 + S^2 = \frac{\alpha(\alpha+1)}{\beta^2}$
		where $S^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - (\bar{x})^2$

Elementary algebra gives

$$\hat{\alpha}_{MM} = \frac{(\bar{x})^2}{S^2} \qquad \hat{\beta}_{MM} = \frac{\bar{x}}{S^2}$$

5. (i) For $\theta > 0$

STEP 1 $L(\theta) = \prod_{i=1}^n f_X(x_i; \theta) = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^n x_i \right)^{\theta-1}$

STEP 2 $\log L(\theta) = n \log \theta + (\theta - 1) \sum_{i=1}^n \log x_i$

STEP 3 $\frac{d}{d\theta} \{\log L(\theta)\} = \frac{n}{\theta} + \sum_{i=1}^n \log x_i = 0 \implies \hat{\theta}_{ML} = -n / \sum_{i=1}^n \log x_i$

STEP 4 $\frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{n}{\theta^2} < 0$ for all θ

Hence

ESTIMATE : $\hat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^n \log x_i}$ ESTIMATOR: $-\frac{n}{\sum_{i=1}^n \log X_i}$

(ii) For $\theta > 0$

STEP 1 $L(\theta) = \prod_{i=1}^n f_X(x_i; \theta) = \prod_{i=1}^n (\theta + 1) x_i^{-(\theta+2)} = (\theta + 1)^n \left(\prod_{i=1}^n x_i \right)^{-(\theta+2)}$

STEP 2 $\log L(\theta) = n \log(\theta + 1) - (\theta + 2) \sum_{i=1}^n \log x_i$

STEP 3 $\frac{d}{d\theta} \{\log L(\theta)\} = \frac{n}{\theta + 1} - \sum_{i=1}^n \log x_i = 0 \implies \hat{\theta}_{ML} = n / \sum_{i=1}^n \log x_i - 1$

STEP 4 $\frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{n}{(\theta + 1)^2} < 0$ for all θ

Hence

ESTIMATE : $\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^n \log x_i} - 1$ ESTIMATOR: $= \frac{n}{\sum_{i=1}^n \log x_i} - 1$

(iii) For $\theta > 0$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i; \theta) = \prod_{i=1}^n \theta^2 x_i \exp\{-\theta x_i\} = \theta^{2n} \left(\prod_{i=1}^n x_i \right) \exp\left\{-\theta \sum_{i=1}^n x_i\right\}$$

$$\text{STEP 2} \quad \log L(\theta) = 2n \log \theta + \sum_{i=1}^n \log x_i - \theta \sum_{i=1}^n x_i$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\theta)\} = \frac{2n}{\theta} - \sum_{i=1}^n x_i = 0 \quad \implies \quad \hat{\theta}_{ML} = \frac{2n}{\sum_{i=1}^n x_i}$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{2n}{\theta^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{2n}{\sum_{i=1}^n x_i} \quad \text{ESTIMATOR: } = \frac{2n}{\sum_{i=1}^n X_i}$$

(iv) Because of the constraint in the pdf that $x \leq \theta$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i; \theta) = \begin{cases} \prod_{i=1}^n 2\theta^2 x_i^{-3} = 2^n \theta^{2n} \left(\prod_{i=1}^n x_i^{-3} \right) & \theta \leq x_1, \dots, x_n \\ 0 & \text{otherwise} \end{cases}$$

$$\text{STEP 2} \quad \log L(\theta) = n \log 2 + 2n \log \theta - 3 \sum_{i=1}^n \log x_i$$

At this point we note that the likelihood is monotonically increasing in θ , and hence the likelihood is maximized when θ is as **large** as possible but so that the constraint $\theta \leq x_1, \dots, x_n$ is still satisfied, hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \min\{x_1, \dots, x_n\} \quad \text{ESTIMATOR: } \min\{X_1, \dots, X_n\}$$

(v) For $\theta > 0$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i; \theta) = \prod_{i=1}^n \left(\frac{\theta}{2} \right) \exp\{-\theta |x_i|\} = 2^{-n} \theta^n \exp\left\{-\theta \sum_{i=1}^n |x_i|\right\}$$

$$\text{STEP 2} \quad \log L(\theta) = -n \log 2 + n \log \theta - \theta \sum_{i=1}^n |x_i|$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\theta)\} = \frac{n}{\theta} - \sum_{i=1}^n |x_i| = 0 \quad \implies \quad \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^n |x_i|}$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^n |x_i|} \quad \text{ESTIMATOR : } \frac{n}{\sum_{i=1}^n |X_i|}$$

(vi) Because of the constraint in the pdf that $\theta_1 \leq x \leq \theta_2$

$$\text{STEP 1} \quad L(\theta_1, \theta_2) \prod_{i=1}^n f_X(x_i; \theta) = \begin{cases} \prod_{i=1}^n \frac{1}{(\theta_2 - \theta_1)} = \frac{1}{(\theta_2 - \theta_1)^n} & \theta_1 \leq x_1, \dots, x_n \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{STEP 2} \quad \log L(\theta_1, \theta_2) = -n \log(\theta_2 - \theta_1)$$

At this point we note that the likelihood is monotonically increasing in θ_1 and monotonically decreasing in θ_2 , and hence the likelihood is maximized when θ_1 is as **large** as possible and when θ_2 is as **small** as possible, but so that the constraint $\theta_1 \leq x_1, \dots, x_n \leq \theta_2$ (and $\theta_2 \geq \theta_1$) is still satisfied, hence

$$\begin{array}{ll} \text{ESTIMATE :} & \hat{\theta}_{1ML} = \min \{x_1, \dots, x_n\} \\ & \hat{\theta}_{2ML} = \max \{x_1, \dots, x_n\} \end{array} \quad \begin{array}{ll} \text{ESTIMATOR:} & \theta_1 = \min \{X_1, \dots, X_n\} \\ & \theta_2 = \max \{X_1, \dots, X_n\} \end{array}$$

(vii) Noting the constraint in the pdf that $x \geq \theta_2$, we have

$$\text{STEP 1} \quad L(\theta_1, \theta_2) = \prod_{i=1}^n f_X(x_i; \theta) = \prod_{i=1}^n \theta_1 \theta_2^{\theta_1} x_i^{-(\theta_1+1)} = \theta_1^n \theta_2^{n\theta_1} \left(\prod_{i=1}^n x_i \right)^{-(\theta_1+1)}$$

$$0 \leq x_1, \dots, x_n \leq \theta_2$$

$$\text{STEP 2} \quad \log L(\theta_1, \theta_2) = n \log \theta_1 + n\theta_1 \log \theta_2 - (\theta_1 + 1) \sum_{i=1}^n \log x_i$$

$$\text{STEP 3} \quad \frac{\partial}{\partial \theta_1} \{\log L(\theta_1, \theta_2)\} = \frac{n}{\theta_1} + n \log \theta_2 - \sum_{i=1}^n \log x_i = 0$$

$$\implies \hat{\theta}_{1ML} = \frac{n}{\sum_{i=1}^n \log x_i - n \log \hat{\theta}_{2ML}}$$

$$\frac{\partial}{\partial \theta_2} \{\log L(\theta_1, \theta_2)\} = \frac{n\theta_1}{\theta_2}$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when θ_2 is as **large** as possible, that is, when $\hat{\theta}_{2ML} = \min \{x_1, \dots, x_n\}$. Hence

$$\hat{\theta}_{1ML} = \frac{n}{\left[\sum_{i=1}^n \log x_i - n \log \{\min \{x_1, \dots, x_n\}\} \right]}$$

ESTIMATES

$$\hat{\theta}_{2ML} = \min \{x_1, \dots, x_n\}$$

$$\theta_1 = \frac{n}{\left[\sum_{i=1}^n \log x_i - n \log \{\min \{X_1, \dots, X_n\}\} \right]}$$

ESTIMATORS:

$$\theta_2 = \min \{X_1, \dots, X_n\}$$

5. Follow the four step procedure that can be summarized as follows: for an observed random sample x_1, \dots, x_n from a distribution represented by mass/density function $f_X(x; \theta)$

STEP 1 : Form the likelihood function $L(\theta)$

$$L(\theta) = \prod_{i=1}^n f_X(x_i; \theta)$$

STEP 2 Take (natural) log to obtain $\log L(\theta)$

STEP 3: Find the value of θ at which $\log L(\theta)$ (and hence $L(\theta)$) is maximized within the parameter space Θ by differentiation

STEP 4: Check the **maximum** value has been found.

Formally, we define the **maximum likelihood estimate** of θ , $\hat{\theta}_{ML}$, as

$$\hat{\theta}_{ML} = \arg \max_{\theta} L(\theta)$$

Hence, for the *Poisson*(λ) case

$$\text{STEP 1} \quad L(\lambda) = \prod_{i=1}^n f_X(x_i; \lambda) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\left(\prod_{i=1}^n x_i! \right)}$$

$$\text{STEP 2} \quad \log L(\lambda) = - \sum_{i=1}^n \log x_i! - n\lambda + \left(\sum_{i=1}^n x_i \right) \log \lambda$$

$$\text{STEP 3} \quad \frac{d}{d\lambda} \{ \log L(\lambda) \} = -n + \left(\sum_{i=1}^n x_i \right) \frac{1}{\lambda} = 0 \quad \implies \quad \hat{\lambda}_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

$$\text{STEP 4} \quad \frac{d^2}{d\lambda^2} \{ \log L(\lambda) \} = - \left(\sum_{i=1}^n x_i \right) \frac{1}{\lambda^2} < 0 \quad \text{for all } \lambda$$

Therefore

$$\text{ESTIMATE : } \hat{\lambda}_{ML} = \bar{x} \quad \text{ESTIMATOR: } \bar{X}$$

and from question 1 on this sheet, we know that if $T_1 = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ then $E_{f_{T_1}} [T_1] = \lambda$ and T_1 is unbiased. Now, if

$$\tau = \tau(\lambda) = e^{-\lambda} \quad \text{so that} \quad \lambda = -\log \tau$$

and we can reformulate the likelihood in terms of τ , giving

$$\log L(\tau) = - \sum_{i=1}^n \log x_i! + n \log \tau + \left(\sum_{i=1}^n x_i \right) \log(-\log \tau)$$

and

$$\frac{d}{d\lambda} \{ \log L(\tau) \} = \frac{\left(\sum_{i=1}^n x_i \right)}{-\tau \log \tau} + \frac{n}{\tau} = 0 \quad \implies \quad \hat{\tau}_{ML} = \exp \left\{ -\frac{1}{n} \sum_{i=1}^n x_i \right\} = \exp \{ -\bar{x} \}$$

which can be shown to be the value that maximizes the likelihood, so that

$$\hat{\tau}_{ML}(\lambda) = \tau(\hat{\lambda}_{ML})$$

6. Noting the constraint in the pdf that $x \geq \eta$, we have

STEP 1
$$L(\lambda, \eta) = \prod_{i=1}^n f_X(x_i; \theta) = \prod_{i=1}^n \lambda \exp\{-\lambda(x_i - \eta)\} = \lambda^n \exp\left\{-\lambda \sum_{i=1}^n (x_i - \eta)\right\}$$

$$\eta \leq x_1, \dots, x_n, \text{ zero otherwise}$$

STEP 2
$$\log L(\lambda, \eta) = n \log \lambda - \lambda \sum_{i=1}^n (x_i - \eta) = n \log \lambda - \lambda \sum_{i=1}^n x_i + n\lambda\eta$$

STEP 3
$$\frac{\partial}{\partial \lambda} \{\log L(\lambda, \eta)\} = \frac{n}{\lambda} + \sum_{i=1}^n x_i - n\eta = 0 \implies \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^n x_i - n\hat{\eta}}$$

$$\frac{\partial}{\partial \eta} \{\log L(\lambda, \eta)\} = n\lambda$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when η is as **large** as possible, that is, when $\hat{\eta} = \min\{x_1, \dots, x_n\}$. Hence

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^n x_i - n \min\{x_1, \dots, x_n\}} = \frac{n}{\sum_{i=1}^n (x_i - \min\{x_1, \dots, x_n\})}$$

ESTIMATES

$$\hat{\eta} = \min\{x_1, \dots, x_n\}$$

$$\lambda = \frac{n}{\sum_{i=1}^n (X_i - \min\{X_1, \dots, X_n\})}$$

ESTIMATORS:

$$\eta = \min\{X_1, \dots, X_n\}$$

7. $X_1, \dots, X_n \sim \text{Exponential}(1/\theta)$ so that $E_{f_{X_i}}[X_i] = \theta$ and hence, using standard mgf techniques, we have

$$X = \sum_{i=1}^n X_i \sim \text{Gamma}\left(n, \frac{1}{\theta}\right) \implies E_{f_X}[X] = \frac{n}{\frac{1}{\theta}} = n\theta$$

so that if $T_1 = \bar{X} = \frac{1}{n}X$ then

$$E_{f_{T_1}}[T_1] = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ .

Now if $Y_1 = \min\{X_1, \dots, X_n\}$, then previous order statistics results give that

$$F_{Y_1}(y) = 1 - \{1 - F_X(y)\}^n = 1 - \left\{1 - \left(1 - e^{-y/\theta}\right)\right\}^n = 1 - e^{-ny/\theta} \quad y > 0$$

so that $Y_1 \sim \text{Exponential}\left(\frac{n}{\theta}\right)$. Hence if $T_2 = nY_1$ then

$$E_{f_{Y_1}}[Y_1] = \frac{\theta}{n} \quad E_{f_{T_2}}[T_2] = n\frac{\theta}{n} = \theta$$

and hence T_2 is an unbiased estimator of θ .

$$f_X(x) = \frac{1}{\theta} \quad \theta - 1 \leq x \leq \theta + 1 \quad F_X(x) = \frac{x - (\theta - 1)}{2} = \frac{x - \theta + 1}{2} \quad \theta - 1 \leq x \leq \theta + 1$$

$E_{f_{X_i}} [X_i] = \theta$ (by integration, or by noting that the pdf is constant and hence symmetric about θ) and hence, using standard expectation techniques, we have that if $T_1 = \bar{X}$

$$E_{f_{T_1}} [T_1] = \frac{1}{n} \sum_{i=1}^n E_{f_{X_i}} [X_i] = \frac{1}{n} \sum_{i=1}^n \theta = \frac{1}{n} n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ .

Now if $Y_1 = \min \{X_1, \dots, X_n\}$ and $Y_n = \max \{X_1, \dots, X_n\}$, then previous order statistics results give that

$$f_{Y_1}(y) = n f_X(y) \{1 - F_X(y)\}^{n-1} = n \frac{1}{2} \left\{ 1 - \frac{y - (\theta - 1)}{2} \right\}^{n-1} = \frac{n}{2} \left\{ \frac{1 + \theta - y}{2} \right\}^{n-1} \quad \theta - 1 \leq y \leq \theta + 1$$

and

$$f_{Y_n}(y) = n f_X(y) \{F_X(y)\}^{n-1} = n \frac{1}{2} \left\{ \frac{y - (\theta - 1)}{2} \right\}^{n-1} = \frac{n}{2} \left\{ \frac{1 - \theta + y}{2} \right\}^{n-1} \quad \theta - 1 \leq y \leq \theta + 1$$

For the expectations,

$$\begin{aligned} E_{f_{Y_1}} [Y_1] &= \int_{\theta-1}^{\theta+1} y \frac{n}{2} \left\{ \frac{1 + \theta - y}{2} \right\}^{n-1} dy \\ &= \frac{n}{2} \int_0^1 ((1 + \theta) - 2t) t^{n-1} 2 dt && \text{setting } t = (1 + \theta - y)/2y = (1 + \theta) - 2t \\ &= (1 + \theta) \int_0^1 n t^{n-1} dt - 2n \int_0^1 t^n dt \\ &= (1 + \theta) - \frac{2n}{n+1} \end{aligned}$$

and

$$\begin{aligned} E_{f_{Y_n}} [Y_n] &= \int_{\theta-1}^{\theta+1} y \frac{n}{2} \left\{ \frac{1 - \theta + y}{2} \right\}^{n-1} dy \\ &= \frac{n}{2} \int_0^1 (2t - (1 - \theta)) t^{n-1} 2 dt && \text{setting } t = (1 - \theta + y)/2y = 2t - (1 - \theta) \\ &= 2 \int_0^1 n t^n dt - (1 - \theta) \int_0^1 n t^{n-1} dt \\ &= \frac{2n}{n+1} - (1 - \theta) \end{aligned}$$

so that if $M = (Y_1 + Y_n)/2$ then by properties of expectations

$$E_{f_M} [M] = \frac{1}{2} E_{f_{Y_1}} [Y_1] + \frac{1}{2} E_{f_{Y_n}} [Y_n] = \left[\frac{1}{2} (1 + \theta) - \frac{n}{n+1} \right] + \left[\frac{n}{n+1} - \frac{1}{2} (1 - \theta) \right] = \theta$$

and hence M is an unbiased estimator for θ

9. (i) For $\lambda > 0$

$$\text{STEP 1} \quad L(\lambda) = \prod_{i=1}^n f_X(x_i; \lambda) = \prod_{i=1}^n \frac{\lambda^2}{\Gamma(2)} x_i \exp\{-\lambda x_i\} = \frac{\lambda^{2n}}{\{\Gamma(2)\}^n} \left(\prod_{i=1}^n x_i \right) \exp\left\{-\lambda \sum_{i=1}^n x_i\right\}$$

$$\text{STEP 2} \quad \log L(\lambda) = 2n \log \lambda - n \log \Gamma(2) + \sum_{i=1}^n \log x_i - \lambda \sum_{i=1}^n x_i$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\lambda)\} = \frac{2n}{\lambda} - \sum_{i=1}^n x_i = 0 \quad \implies \quad \hat{\lambda}_{ML} = \frac{2n}{\sum_{i=1}^n x_i}$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\lambda)\} = -\frac{2n}{\lambda^2} < 0 \quad \text{at } \lambda = \hat{\lambda}_{ML}$$

Hence the estimator is $2n / \left(\sum_{i=1}^n X_i \right)$

(ii) The result from q. 4 can be generalized so that for any function $\tau = \tau(\lambda)$

$$\hat{\tau}_{ML}(\lambda) = \tau(\hat{\lambda}_{ML})$$

which is referred to as the **invariance** property of maximum likelihood estimators. Thus we must have that the ML estimator of $\tau = 1/\lambda$ is

$$T = \frac{1}{2n} \sum_{i=1}^n X_i$$

Now, using mgfs, it is straightforward to show that

$$\sum_{i=1}^n X_i \sim \text{Gamma}(2n, \lambda) \quad T = \frac{1}{2n} \sum_{i=1}^n X_i \sim \text{Gamma}(2n, 2n\lambda)$$

so that

$$E_{f_T}[T] = \frac{2n}{2n\lambda} = \frac{1}{\lambda} \quad \text{Var}_{f_T}[T] = \frac{2n}{(2n\lambda)^2} = \frac{1}{2n\lambda^2} \quad E_{f_T}[T^2] = \frac{1}{2n\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2n+1}{2n\lambda^2}$$

(iii) Using the Central Limit Theorem we have that

$$\frac{\sum_{i=1}^n X_i - \frac{2n}{\lambda}}{\sqrt{\frac{2n}{\lambda^2}}} \xrightarrow{d} Z \sim N(0, 1)$$

so that, dividing through by $2n$

$$\frac{T - \frac{1}{\lambda}}{\sqrt{\frac{1}{2n\lambda^2}}} \xrightarrow{d} Z \sim N(0, 1)$$

and hence, approximately

$$T \sim N\left(\frac{1}{\lambda}, \frac{1}{2n\lambda^2}\right)$$

and as $n \rightarrow \infty$ then variance tends to zero, and hence for $\varepsilon > 0$

$$P\left[\left|T - \frac{1}{\lambda}\right| < \varepsilon\right] \rightarrow 1 \quad T \xrightarrow{p} \frac{1}{\lambda}$$