M2S1 - EXERCISES 4: SOLUTIONS

1. The cdf of X, F_X is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x 4t^3\ dt = x^4 \qquad 0 < x < 1.$$

(a) $Y = X^4$, so $\mathbb{Y} = (0,1)$, and from first principles, for $y \in \mathbb{Y}$,

$$F_Y(y) \;= \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ X^4 \leq y\ ] = \mathrm{P}[\ X \leq y^{1/4}\ ] = F_X(y^{1/4}) = y \qquad \Longrightarrow f_Y(y) = 1 \qquad 0 < y < 1$$

(b) $W = e^X$, so $\mathbb{W} = (1,e)$, and from first principles, for $w \in \mathbb{W}$,

$$F_W(w) \;=\; \mathrm{P}[\ W \leq w\ ] = \mathrm{P}[\ e^X \leq w\ ] = \mathrm{P}[\ X \leq \log w\ ] = F_X(\log w) = (\log w)^4$$

$$\Longrightarrow \quad f_W(w) = \frac{4(\log w)^3}{w} \qquad 1 < w < e$$

(c) $Z = \log X$, so $\mathbb{Z} = (-\infty, 0)$, and from first principles, for $z \in \mathbb{Z}$,

$$F_Z(z) = \mathrm{P}[\ Z \leq z\ ] = \mathrm{P}[\ \log X \leq z\ ] = \mathrm{P}[\ X \leq e^z\ ] = F_X(e^z) = e^{4z} \Longrightarrow f_Z(z) = 4e^{4z} \qquad -\infty < z < 0$$

(d) $U = (X - 0.5)^2$, so $\mathbb{U} = (0, 0.25)$, and from first principles, for $u \in \mathbb{U}$,

$$\begin{aligned} F_U(u) &= \mathrm{P}[\ U \leq u\ ] = \mathrm{P}[\ (X-0.5)^2 \leq u\ ] = \mathrm{P}[\ -\sqrt{u} + 0.5 \leq X \leq \sqrt{u} + 0.5\ ] \\ &= F_X(\sqrt{u} + 0.5) - F_X(-\sqrt{u} + 0.5) = (0.5 + \sqrt{u})^4 - (0.5 - \sqrt{u})^4 \\ \Longrightarrow f_U(u) &= \frac{2}{\sqrt{u}}\left[(0.5+\sqrt{u})^3 + (0.5-\sqrt{u})^3\right] = \frac{1+12u}{2\sqrt{u}} \qquad 0 < u < 0.25 \end{aligned}$$

To find the decreasing function H on $(0,1)$; need $F_V(v) = v,\ 0 < v < 1$, that is, need

$$\begin{aligned} \mathrm{P}[\ V \;&\leq\; v\ ] = \mathrm{P}[\ H(X) \leq v\ ] = v \Longrightarrow \mathrm{P}[\ X \geq H^{-1}(v)\ ] = v \Longrightarrow 1 - \mathrm{P}[\ X < H^{-1}(v)\ ] = v \\ &\Longrightarrow \left\{H^{-1}(v)\right\}^4 = 1 - v \text{ and hence } H(v) = 1 - v^4 \end{aligned}$$

2. We have $f_R(r) = 6r(1-r)$, for $0 < r < 1$, and hence

$$F_R(r) = r^2(3 - 2r) \qquad 0 < r < 1$$

Circumference: $Y = 2\pi R$, so $\mathbb{Y} = (0, 2\pi)$, and from first principles, for $y \in \mathbb{Y}$,

$$\begin{aligned} F_Y(y) &= \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ 2\pi R \leq y\ ] = \mathrm{P}[\ R \leq y/2\pi\ ] = F_R(y/2\pi) = \frac{3y^2}{4\pi^2} - \frac{2y^3}{8\pi^3} \\ \Longrightarrow f_Y(y) &= \frac{6y}{8\pi^3}(2\pi - y) \qquad 0 < y < 2\pi \end{aligned}$$

Area: $Z = \pi R^2$, so $\mathbb{Z} = (0, \pi)$, and from first principles, for $z \in \mathbb{Z}$, recalling that f_R is only positive when $0 < z < \pi$,

$$\begin{aligned} F_Z(z) &= \mathrm{P}[\ Z \leq z\ ] = \mathrm{P}[\ \pi R^2 \leq z\ ] = \mathrm{P}[\ R \leq \sqrt{z/\pi}\ ] = F_R(\sqrt{z/\pi}) = \frac{3z}{\pi} - 2\left\{\frac{z}{\pi}\right\}^{3/2} \\ \Longrightarrow f_Z(z) &= 3\pi^{-3/2}(\sqrt{\pi} - \sqrt{z}) \qquad 0 < z < \pi. \end{aligned}$$

3. By integration

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\ dt = \int_0^x \frac{\alpha}{\beta}\left(\frac{\beta}{\beta+t}\right)^{\alpha+1} dt = \left[-\left(\frac{\beta}{\beta+t}\right)^{\alpha}\right]_0^x = 1-\left(1+\frac{x}{\beta}\right)^{-\alpha} \qquad x>0.$$

If $Y = \log X$, then $\mathbb{Y} = \mathbb{R}$, and

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ \log X \leq y\ ] = \mathrm{P}[\ X \leq e^y\ ] = F_X(e^y) = 1-\left(1+\frac{e^y}{\beta}\right)^{-\alpha}$$

$$\Longrightarrow f_Y(y) = \frac{\alpha}{\beta}e^y\left(\frac{\beta}{\beta+e^y}\right)^{\alpha+1} \qquad y \in \mathbb{R}$$

If $Z = \xi + \theta Y$, then $Y = (Z-\xi)/\theta$, so the density of Z can be found easily using transformation techniques

$$f_Z(z) = \frac{\alpha}{\beta}e^{(z-\xi)/\theta}\left(\frac{\beta}{\beta+e^{(z-\xi)/\theta}}\right)^{\alpha+1}\frac{1}{\theta} \qquad z \in \mathbb{R}$$

4. Easy to see that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, with $\mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$, so X and Y are independent, where

$$f_X(x) = \sqrt{c}\,\frac{2^x}{x!} \qquad f_Y(y) = \sqrt{c}\,\frac{2^y}{y!} \qquad \text{and} \qquad \sum_{x=0}^{\infty} f_X(x) = 1 \Longrightarrow \sqrt{c} = e^{-2}$$

(marginal mass functions must have identical forms as joint mass function is symmetric in x and y) as the summation is identical to the power series expansion of e^z at $z=2$ if $\sqrt{c} = e^{-2}$.

5. $F_{X,Y}$ is continuous and non decreasing in x and y, and

$$\lim_{x\longrightarrow -\infty} F_{X,Y}(x,y) = \lim_{y\longrightarrow -\infty} F_{X,Y}(x,y) = 0 \qquad \lim_{x,y\longrightarrow \infty} F_{X,Y}(x,y) = 1$$

so $F_{X,Y}$ is a valid cdf, and

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial t_1 \partial t_2}\left\{F_{X,Y}(t_1,t_2)\right\}_{t_1=x,t_2=y} = \frac{e^{-x}}{\pi(1+y^2)} = f_X(x)f_Y(y)$$

so as $\mathbb{X}^{(2)} = \mathbb{R}^+ \times \mathbb{R}$, X and Y are independent.

6. (i) If $\mathbb{X}^{(2)} = (0,1) \times (0,1)$ is the (joint) range of vector random variable (X,Y). We have

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0<x<1,\ 0<y<1$$

so that

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \qquad \text{and} \qquad \mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are the ranges of X and Y respectively, and

$$f_X(x) = c_1 x \qquad \text{and} \qquad f_Y(y) = c_2(1-y) \tag{1}$$

for some constants satisfying $c_1c_2 = c$. Hence, the two conditions for independence are satisfied in (1), and X and Y are independent.

(ii) We must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = 1 \qquad \therefore \qquad c^{-1} = \int_0^1\int_0^1 x(1-y)\ dxdy = 1$$

and as

$$\int_0^1\int_0^1 x(1-y)\ dxdy = \left\{\int_0^1 x\ dx\right\}\left\{\int_0^1 (1-y)\ dy\right\} = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

we have $c = 4$.

(iii) We have $A = \{(x,y)\ :\ 0 < x < y < 1\}$, and hence, recalling that the joint density is only non-zero when $x < y$, we first fix a y and integrate dx on the range $(0, y)$, and then integrate dy on the range $(0, 1)$, that is

$$P[\ X < Y\ ] = \int\int_A f_{X,Y}(x,y)\ dxdy = \int_0^1 \left\{\int_0^y 4x(1-y)\ dx\right\} dy = \int_0^1 \left\{\int_0^y x\ dx\right\} 4(1-y)\ dy$$

$$= \int_0^1 2y^2(1-y)\ dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_0^1 = \frac{1}{6}$$

7. The joint pdf of X and Y is given by

$$f_{X.Y}(x,y) = 24xy \qquad x > 0,\ y > 0,\ x + y < 1$$

and zero otherwise, the marginal pdf f_X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{1-x} 24xy\ dy = 24x\left[\frac{y^2}{2}\right]_0^{1-x} = 12x(1-x)^2 \qquad 0 < x < 1$$

as the integrand is only non-zero when $0 < x + y < 1 \Longrightarrow 0 < y < 1 - x$ for fixed x

8. First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \leq 1$ and $y > 1$. The marginal distributions are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_{1/x}^{x} \frac{1}{2x^2y}\ dy = \frac{1}{2x^2}(\log x - \log(1/x)) = \frac{\log x}{x^2} \qquad 1 \leq x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \begin{cases} \displaystyle\int_{1/y}^{\infty} \frac{1}{2x^2y}\ dx = \frac{1}{2} & 0 \leq y \leq 1 \\ \displaystyle\int_{y}^{\infty} \frac{1}{2x^2y}\ dx = \frac{1}{2y^2} & 1 \leq y \end{cases}$$

Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{x^2y} & 1/y \leq x \text{ if } 0 \leq y \leq 1 \\ \dfrac{y}{x^2} & y \leq x \text{ if } 1 \leq y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y\log x} \qquad 1/x \leq y \leq x \text{ if } x \geq 1$$

Marginal expectation of Y;

$$E_{f_Y}[\ Y\ ] = \int_{-\infty}^{\infty} y f_Y(y)\ dy = \int_0^1 \frac{y}{2}\ dy + \int_1^{\infty} \frac{1}{2y}\ dy = \infty$$

as the second integral is divergent.