EXPECTATION FACTSHEET

- Random variable X
- Mass/density function f_X with support $\mathbb{X}$.
- **Expectation**

$$E_{f_X}[X] = \begin{cases} \displaystyle\sum_{x \in \mathbb{X}} x f_X(x) & X \text{ discrete} \\ \displaystyle\int_{-\infty}^{\infty} x f_X(x) dx \quad = \int_{\mathbb{X}} x f_X(x) dx & X \text{ continuous} \end{cases}$$

In the discrete case, if X only takes values on (a subset of) the integers, we can also write

$$E_{f_X}[X] = \sum_{x=-\infty}^{\infty} x f_X(x)$$

- **Extension**: Let g be a real-valued function whose domain includes X. Then

$$E_{f_X}[g(X)] = \begin{cases} \displaystyle\sum_{x=-\infty}^{\infty} g(x) f_X(x) & X \text{ discrete} \\ \displaystyle\int_{-\infty}^{\infty} g(x) f_X(x) dx & X \text{ continuous} \end{cases}$$

Note that the sum/integral may be **divergent**, so that the expectation is **not finite**.

PROPERTIES

1. **Linearity:** Let g and h be real-valued functions whose domains include $\mathbb{X}$, and let a and b be constants.

$$\begin{aligned} E_{f_X}[ag(X)+bh(X)] &= \int [ag(x)+bh(x)]f_X(x)dx \\ &= a\int g(x)f_X(x)dx + b\int h(x)f_X(x)dx \\ &= aE_{f_X}[g(X)] + bE_{f_X}[h(X)] \end{aligned}$$

   Hence, for example,

$$E_{f_X}[aX+b] = aE_{f_X}[X] + b.$$

2. Consider $g(x) = (x - \mathrm{E}_{f_X}[X])^2 = (x-\mu)^2$, say. Then

$$\begin{aligned} E_{f_X}[g(X)] &= \int (x-\mu)^2 f_X(x)dx = \int x^2 f_X(x)dx - 2\mu\int x f_X(x)dx + \mu^2\int f_X(x)dx \\ &= \int x^2 f_X(x)dx - 2\mu^2 + \mu^2 = \int x^2 f_X(x)dx - \mu^2 \\ &= E_{f_X}[X^2] - \{E_{f_X}[X]\}^2 \end{aligned}$$

   Thus

   (i) **Variance:** $Var_{f_X}[X] = E_{f_X}[X^2] - \{E_{f_X}[X]\}^2$

   (ii) **Standard deviation** $\sqrt{Var_{f_X}[X]}$

3. Consider $g(x) = x^k$ for $k = 1, 2, ...$ Then in the continuous case

$$E_{f_X}[g(X)] = E_{f_X}[X^k] = \int x^k f_X(x)dx,$$

   and $\mathrm{E}_{f_X}[X^k]$ is the kth **moment** of the distribution.

4. Consider $g(x) = (x-\mu)^k$ for $k = 1, 2, ...$ Then

$$E_{f_X}[g(X)] = E_{f_X}[(X-\mu)^k] = \int (x-\mu)^k f_X(x)dx,$$

   and $E_{f_X}[(X-\mu)^k]$ is the kth **central moment** of the distribution.

5. Consider $g(x) = aX + b$. Then

$$\begin{aligned} Var_{f_X}[g(X)] = E_{f_X}[(aX+b-E_{f_X}[aX+b])^2] &= E_{f_X}[(aX+b-aE_{f_X}[X]-b)^2] \\ &= E_{f_X}[(a^2(X-E_{f_X}[X])^2] \\ &= a^2 Var_{f_X}[X]. \end{aligned}$$

   so

$$Var_{f_X}[aX+b] = a^2 Var_{f_X}[X].$$

6. Consider $g(x) = e^{tx}$, for constant $t \in (-h, h)$ for some $h > 0$, and

$$M_X(t) = E_{f_X}[g(X)] = E_{f_X}\left[e^{tX}\right].$$

   Then $M_X(t)$ is the **moment generating function**.