DISTRIBUTIONS FACTSHEET

DISCRETE DISTRIBUTIONS

Models based on an independent sequence of identical binary trials with success probability θ.

- **BERNOULLI** X is the total number of successes in one trial.
- **BINOMIAL** X is the total number of successes in n trials.
- **GEOMETRIC** X is the total number of trials required to obtain **one** success.
- **NEGATIVE BINOMIAL** X is the total number of trials required to obtain **n** successes. Alternative form given by considering $Y = X - n$, to give a distribution on $\{0, 1, 2, \ldots\}$.
- **POISSON** X is the count of the number of events in a given (continuous) time interval. The Poisson distribution is obtained as the limiting form of the $Binomial(n, \theta)$ distribution, with $\lambda = n\theta$ held fixed.

Connections:

- Bernoulli/Binomial

$$X_1, \ldots, X_n \sim Bernoulli(\theta) \qquad \Rightarrow \qquad Y = \sum_{i=1}^{n} X_i \sim Binomial(n, \theta)$$

- Geometric/Negative Binomial

$$X_1, \ldots, X_n \sim Geometric(\theta) \qquad \Rightarrow \qquad Y = \sum_{i=1}^{n} X_i \sim NegBinomial(n, \theta)$$

- Binomial/Poisson

$$X_n \sim Binomial(n, \theta) \longrightarrow X \sim Poisson(\lambda)$$

  where $\lambda = n\theta$ is held fixed and $n \longrightarrow \infty$.

- Negative Binomial/Poisson

$$X_n \sim NegBinomial(n, \theta) \qquad Y_n = X_n - n \longrightarrow Y \sim Poisson(\lambda)$$

  where $\lambda = n(1 - \theta)$ is held fixed and $n \longrightarrow \infty$.

Summations of Independent RVs:

- Binomial

$$\left.\begin{array}{r} X \sim Binomial(m, \theta) \\ Y \sim Binomial(n, \theta) \end{array}\right\} \qquad \Rightarrow \qquad T = X + Y \sim Binomial(m + n, \theta)$$

- Negative Binomial

$$\left.\begin{array}{r} X \sim NegBinomial(m, \theta) \\ Y \sim NegBinomial(n, \theta) \end{array}\right\} \qquad \Rightarrow \qquad T = X + Y \sim NegBinomial(m + n, \theta)$$

- Poisson

$$\left.\begin{array}{r} X \sim Poisson(\lambda_X) \\ Y \sim Poisson(\lambda_Y) \end{array}\right\} \qquad \Rightarrow \qquad T = X + Y \sim Poisson(\lambda_X + \lambda_Y)$$

CONTINUOUS DISTRIBUTIONS

- Distributions on $\mathbb{R}^+$

  Begin with $U \sim Uniform(0,1)$:

  - ► $X = -\dfrac{1}{\lambda}\log U \sim Exponential(\lambda)$, for $\lambda > 0$.
  - ► $Y = X^{1/\alpha} \sim Weibull(\alpha, \lambda)$, for $\alpha > 0$.
  - ► If $X_1, \dots, X_n \sim Exponential(\lambda)$, independent, then $Y = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$.
  - ► If $X \sim Gamma(\alpha_X, \lambda)$ and $Y \sim Gamma(\alpha_Y, \lambda)$ are independent, then

$$T = X + Y \sim Gamma(\alpha_X + \alpha_Y, \lambda)$$

- Poisson Process links

  Consider events occurring independently at a constant rate λ in continuous time. Let

$$\begin{aligned} X(t,s) &\equiv \text{number of events occurring in interval } [t,s) \\ X_i &\equiv \text{time between event } i-1 \text{ and event } i \\ Y_n &\equiv \text{time of event } n \end{aligned}$$

  - ► $X(t,s) \sim Poisson(\lambda(s-t))$
  - ► $X(0,t)$ and $X(t,s)$ are independent for $s > t$.
  - ► $X_i \sim Exponential(\lambda)$, with $X_1, X_2, \dots$ independent.
  - ► $Y_n = \sum_{i=1}^{n} X_i \sim Gamma(n, \lambda)$.

- Distributions on $\mathbb{R}$: The Normal distribution and connections

  - ► Suppose $X \sim N(0,1)$. Then $Y = \mu + \sigma X \sim N(\mu, \sigma^2)$.
  - ► Suppose $X \sim N(0,1)$. Then $Y = X^2 \sim Gamma(1/2, 1/2) \equiv Chisquared(1)$.
  - ► If $X_1, X_2 \sim N(0,1)$, and $V \sim Chisquared(\nu)$ are all independent, then

$$T_1 = \frac{X_1}{X_2} \sim Cauchy \qquad T_2 = \frac{X_1}{\sqrt{V/\nu}} \sim Student(\nu)$$

  - ► If $V_1 \sim Chisquared(\nu_1)$ and $V_2 \sim Chisquared(\nu_2)$ are independent, then

$$T_3 = \frac{V_1/\nu_1}{V_2/\nu_2} \sim Fisher(\nu_1, \nu_2)$$

  - ► If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$Y = X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- Distribution on $(0,1)$: The Beta distribution

  - ► If $X_1 \sim Gamma(\alpha_1, \beta)$ and $X_2 \sim Gamma(\alpha_2, \beta)$ are independent, then

$$V = \frac{X_1}{X_1 + X_2} \sim Beta(\alpha_1, \alpha_2)$$