M2S1 - EXERCISES 6

Connections between Distributions

1. The joint pdf $f_{X,Y}$ of positive random variables X and Y is specified as

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$$

where $X|Y = y \sim Exponential(y)$ and $Y \sim Gamma(\alpha, \beta)$. Identify the marginal distribution of X.

2. *The Bivariate Normal Distribution*: Suppose that X_1 and X_2 are i.i.d $Normal(0,1)$ random variables. Let random variables Y_1 and Y_2 be defined by

$$\begin{aligned} Y_1 &= \mu_1 + \sigma_1\sqrt{1-\rho^2}X_1 + \sigma_1\rho X_2 \\ Y_2 &= \mu_2 + \sigma_2 X_2 \end{aligned} \qquad \text{or equivalently} \qquad \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \sigma_1\sqrt{1-\rho^2} & \sigma_1\rho \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

for positive constants σ_1 and σ_2, and $|\rho| < 1$. Find the joint pdf of (Y_1, Y_2).

Show that, marginally for $i = 1, 2$, $Y_i \sim Normal\left(\mu_i, \sigma_i^2\right)$, and that conditionally

$$\begin{aligned} Y_1|Y_2 = y_2 &\sim Normal\left(\mu_1 + \frac{\rho\sigma_1}{\sigma_2}(y_2 - \mu_2), \sigma_1^2\left(1-\rho^2\right)\right) \\ Y_2|Y_1 = y_1 &\sim Normal\left(\mu_2 + \frac{\rho\sigma_2}{\sigma_1}(y_1 - \mu_1), \sigma_2^2\left(1-\rho^2\right)\right) \end{aligned}.$$

Find the correlation of Y_1 and Y_2.

3. Suppose that U_1 and U_2 are i.i.d $Uniform(0,1)$ random variables. Let random variables Z_1 and Z_2 be defined by

$$\begin{aligned} Z_1 &= \sqrt{-2\log U_1}\cos(2\pi U_2) \\ Z_2 &= \sqrt{-2\log U_1}\sin(2\pi U_2) \end{aligned}$$

(log is the natural logarithm). Find the joint pdf of (Z_1, Z_2).

4. Suppose that U is a $Uniform(0,1)$ random variable. Find the distribution of

$$X = -\beta \log U.$$

Suppose that an unlimited sequence of $Uniform(0,1)$ random variables is available. Using the distribution of X, and results from lectures, describe how to generate

(i) a $Gamma(k, \lambda)$ random variable, for integer $k \geq 0$.

(ii) a realization of a *Poisson process* with rate μ.

(iii) a $Chisquare(\nu) \equiv Gamma\left(\frac{\nu}{2}, \frac{1}{2}\right)$ random variable, where ν is a positive, real parameter.

(iv) a $Student(n)$ random variable, where n is a positive integer parameter.

Use the results from question 3., and results given in lectures and the printed notes.