M2S1 - EXERCISES 4

Univariate Transformations

1. Suppose that X is a continuous random variable with density function given by

$$f_X(x) = 4x^3 \qquad 0 < x < 1$$

and zero otherwise. Find the density functions of the following random variables

(a) $Y = X^4$ (b) $W = e^X$ (c) $Z = \log X$ (d) $U = (X - 0.5)^2$

Find the monotonic **decreasing** function H such that the random variable V, defined by $V = H(X)$, has a density function that is constant on the interval $(0, 1)$, and zero otherwise.

2. The measured radius of a circle, R, is a continuous random variable with density function given by

$$f_R(r) = 6r(1 - r) \qquad 0 < r < 1$$

and zero otherwise. Find the density functions of the circumference and the area of the circle.

3. (Harder) Suppose that X is a continuous random variable with density function given by

$$f_X(x) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\alpha+1)} \qquad x > 0$$

for constants $\alpha, \beta > 0$, and zero otherwise. Find the density function and cdf of the random variable defined by $Y = \log X$, and the density function of the random variable defined by $Z = \xi + \theta Y$.

Discrete And Continuous Multivariate Distributions

4. Suppose that X and Y are discrete random variables with joint mass function given by

$$f_{X,Y}(x, y) = c\frac{2^{x+y}}{x!y!} \qquad x,\ y = 0, 1, 2, \ldots,$$

and zero otherwise, for some constant c.

(i) Find the value of c, and the marginal mass functions of X and Y.

(ii) Prove that X and Y are **independent** random variables, that is, that

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

for all $x, y = 0, 1, \ldots$.

5. Continuous random variables X and Y have joint cdf, $F_{X,Y}$ defined by

$$F_{X,Y}(x, y) = \left(1 - e^{-x}\right)\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1} y\right) \qquad x > 0, -\infty < y < \infty$$

with

$$F_{X,Y}(x, y) = 0 \qquad x \leq 0.$$

Find the joint pdf, $f_{X,Y}$. Are X and Y independent ? Justify your answer.

6. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

and zero otherwise for some constant c. Are X and Y independent random variables ?

Find the value of c, and, for the set $A \equiv \{(x,y) : 0 < x < y < 1\}$, the probability

$$P[X < Y] = \int\!\!\int_A f_{X,Y}(x,y)\ dxdy$$

7. Suppose that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = 24xy \qquad x > 0,\ y > 0,\ x + y < 1$$

and zero otherwise. Find the marginal pdf of X, f_X.

Hint: Sketch the region on which the joint density is non-zero; remember that the integrand is only non-zero for some part of the integral range

8. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x,y) = \frac{1}{2x^2 y} \qquad 1 \leq x < \infty, 1/x \leq y \leq x$$

and zero otherwise.

Derive

(i) the **marginal** pdf of X,

(ii) the **marginal** pdf of Y,

(iii) the **conditional** pdf of X given $Y = y$,

(iv) the **conditional** pdf of Y given $X = x$.

Find the (marginal) expectation of Y, $E_{f_Y}[Y]$.