M2S1 - EXERCISES 3

Continuous Probability Distributions and Expectations

1. Show that the function, F_X, defined for $x \in \mathbb{R}$ by

$$F_X(x) = c \exp\left\{-e^{-\lambda x}\right\}$$

is a valid cdf for a continuous random variable X for a specific choice of constant c where parameter $\lambda > 0$. Find the pdf, f_X associated with this cdf.

Now consider the function $f_X(x) = cg(x)$ for some constant $c > 0$, with g defined by

$$g(x) = \frac{|x|}{(1+x^2)^2} \qquad x \in \mathbb{R}$$

Show that $f_X(x)$ is a valid pdf for a continuous random variable X with range $\mathbb{X} = \mathbb{R}$, and find the cdf, F_X, and the expected value of X, $E_{f_X}[X]$, associated with this pdf.

2. Let X be a continuous random variable with range $\mathbb{X} = \mathbb{R}^+$, pdf f_X and cdf F_X. By writing the expectation in its integral definition form on the left hand side, and changing the order of integration show that

$$E_{f_X}[X] = \int_0^\infty [1 - F_X(x)]\ dx$$

Using an identical approach, show also that for integer $r \geq 1$,

$$E_{f_X}[X^r] = \int_0^\infty rx^{r-1}[1 - F_X(x)]\ dx$$

Find a similar expression for random variables for which $\mathbb{X} = \mathbb{R}$.

3. (Harder) Suppose that continuous random variables X_1 and X_2 both with range $\mathbb{X} = \mathbb{R}^+$ have pdfs f_1 and f_2 respectively such that

$$\begin{aligned} f_1(x) &= cx^{-1}\exp\left\{-(\log(x))^2/2\right\} \qquad x > 0 \\ f_2(x) &= f_1(x)\left[1 + \sin(2\pi \log x)\right] \qquad x > 0 \end{aligned}$$

and $f_1(x) = f_2(x) = 0$ for $x \leq 0$. If, for $r = 1, 2, ...$, $E_{f_1}[X_1^r] = \exp\left\{r^2/2\right\}$, show that

$$E_{f_2}[X_2^r] = \exp\left\{r^2/2\right\}$$

Hint: write out the integral for $E_{f_2}[X_2^r]$, and then make a transformation $t = \log(x)$ in the integral. Then complete the square.

4. Suppose that X is a continuous random variable with range $\mathbb{R}$ and pdf given by

$$f_X(x) = \alpha^2 x \exp\{-\alpha x\} \qquad x \geq 0$$

and zero otherwise, for parameter $\alpha > 0$. Find the cdf of X, F_X, and hence show that, for any positive value m,

$$P[X \geq m] = (1 + \alpha m)\exp\{-\alpha m\}$$

Now find $E_{f_X}[X]$. If the expected value of X is increased to $2/\beta$ (for $0 < \beta < \alpha$), find the associated change in $P[X \geq m]$.

Generating Functions

5. The continuous random variable Z with range $\mathbb{Z} = \mathbb{R}$ has pdf given by

$$f_z(z) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}z^2\right\} \qquad z \in \mathbb{R}$$

(a) Find the mgf of random variable Z, the pdf and the mgf of random variable X where

$$X = \mu + \frac{1}{\lambda}Z.$$

for parameters μ and $\lambda > 0$.

Find the expectation of X, and the expectation of the function $g(X)$ where $g(x) = e^x$.

(b) Suppose now Y is the random variable defined in terms of X by $Y = e^X$ Find the pdf of Y, and show that the expectation of Y is

$$\exp\left\{\mu + \frac{1}{2\lambda^2}\right\}$$

(c) Finally, let random variable T be defined by $T = Z^2$. Find the pdf and mgf of Z.

6. Suppose that random variable X has mgf M_X given by

$$M_X(t) = \frac{1}{8}e^t + \frac{2}{8}e^{2t} + \frac{5}{8}e^{3t}$$

Find the probability distribution, and the expectation and variance of X (*hint: consider G_X, and its definition).*

7. Suppose that random variable X has mgf given by

$$M_X(t) = \left(1 - \theta + \theta e^t\right)^n$$

for some θ, where $0 \le \theta \le 1$. Obtain a power series expansion for $M_X(t)$, and hence identify $\mathrm{E}_{f_X}[X^r]$ for $r = 1, 2, 3, \ldots$.

8. Suppose that X is a continuous random variable with pdf

$$f_X(x) = \exp\{-(x+2)\} \qquad -2 < x < \infty$$

Find the mgf of X, and **hence** find the expectation and variance of X.

9. Suppose that X is a random variable with mass function/pdf f_X and mgf M_X. The *cumulant generating function* of X, K_X, is defined by $K_X(t) = \log[M_X(t)]$. Prove that

$$\frac{d}{dt}\{K_X(t)\}_{t=0} = E_{f_X}[X] \qquad\qquad \frac{d^2}{dt^2}\{K_X(t)\}_{t=0} = Var_{f_X}[X]$$