M2S1 - EXERCISES 1

Conditional Probability, The Theorem of Total Probability and Bayes Theorem.

1. For events A and B in sample space Ω, under what conditions does the equation

$$P(A) = P(A|B) + P(A|B')$$

hold ?

2. A biased coin is tossed repeatedly, with tosses mutually independent; the probability of the coin showing Heads on any toss is p. Let H_n be the event that an even number of Heads have been obtained after n tosses, let $p_n = P(H_n)$, and define $p_0 = 1$. By conditioning on H_{n-1} and using the **Theorem of Total Probability**, show that, for $n \geq 1$,

$$p_n = (1 - 2p)p_{n-1} + p. \qquad (1)$$

Find a solution to this difference equation, valid for all $n \geq 0$, of the form $p_n = A + B\lambda^n$, where A, B and λ are constants to be identified. Prove that if $p < 1/2$ then $p_n > 1/2$ for all $n \geq 1$, and find the limiting value of p_n as $n \longrightarrow \infty$. Is this limit intuitively reasonable ?

3. A simple model for weather forecasting involves classifying days as either Fine or Wet, and then assuming that the weather on a given day will be the same as the weather on the preceding day with probability p. Suppose that the probability of fine weather on day indexed 1 (say Jan 1st) is denoted θ. Let θ_n denote the probability that day indexed n is Fine. For $n = 2, 3, ...$, find a difference equation for θ_n similar to that in equation (1) in Problem 2 above, and use this difference equation to find θ_n explicitly as a function of n, p and θ. Find the limiting value of θ_n as $n \longrightarrow \infty$.

4. (a) Consider two coins, of which one is normal and the other has a Head on both sides. A coin is selected and tossed n times with tosses mutually independent. Evaluate the conditional probability that the selected coin is normal, given that the first n tosses are Heads. *[You will need to use the Binomial distribution from M1S.]*

(b) Now consider two coins, of which one is normal and the other is biased so that the probability of obtaining a Head is $p > 1/2$. Again, one of the coins is selected and tossed n times. Let E be the event that the n tosses result in k Heads and $n - k$ Tails, and let F be the event that the coin is fair. Find expressions for $P(E)$ and $P(F|E)$.

5. The probability that a tree has n flowers is given by $(1 - p)p^n$ for $n = 0, 1, 2, ...$ Each flower has probability 2/3 of being pollinated and producing fruit, and each fruit has probability of 1/4 of not ripening fully. It can be assumed that each developmental stage is independent of the others.

(a) Deduce that the probability of a flower producing a ripe fruit is 1/2.
(b) Given that a tree bears r ripe fruit, calculate the conditional probability that it originally had n flowers. *[You will need to use the Negative Binomial expansion.]*

6. A company is to introduce mandatory drug testing for its employees. The test used is very accurate, in that it it gives a correct positive test (detects drugs when they are present in a blood sample) with probability 0.99, and a correct negative test (does not detect drugs when they are not present) with probability 0.98. If an individual tests positive on the first test, a second blood sample is tested. It is assumed that only 1 in 5000 employees actually does provide a blood sample with drugs present.

What is the probability that the presence of drugs in a blood sample is **detected correctly**, given
(i) a positive result on the first test (before the second test is carried out)
(ii) a positive result on both first and second tests.

Assume that the results of tests are conditionally independent, that is, independent given the presence or absence of drugs in the sample.