M2S1 - ASSESSED COURSEWORK 3

To be handed in no later than Wednesday, 14th December, 2.00pm.

Please hand in to the Mathematics General Office.

(a) Vector random variable $\underset{\sim}{X} = (X_1, X_2, X_3)^\mathsf{T}$ has a multivariate normal distribution with pdf given by

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \left(\frac{1}{2\pi}\right)^{3/2} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\underset{\sim}{x}^\mathsf{T}\Sigma^{-1}\underset{\sim}{x}\right\}$$

where variance-covariance matrix Σ is the 3×3 matrix defined as

$$\Sigma = \begin{bmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}.$$

(i) Find the pdf for vector random variable $\underset{\sim}{Y}$ defined by $\underset{\sim}{Y} = A\underset{\sim}{X}$ for matrix A given by

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & -1 & 1 \end{bmatrix}.$$

[4 MARKS]

(ii) Find the correlation between Y_1 and Y_2.

[4 MARKS]

(b) Suppose that random variables $X_1, X_2, \ldots$ are independent and identically distributed random variables with probability distribution described by cdf F_X.

(i) Suppose that $X_1, X_2, \ldots, X_n$ are to be observed. Let $Y_n(x)$ be the discrete random variable defined as the number (out of n) of the Xs that are no greater than x, for fixed $x \in \mathbb{R}$.

Find the probability distribution of $Y_n(x)$, and state the expectation and variance of $Y_n(x)$.

Hint: consider the events "$X_i \leq x$" for $i = 1, \ldots, n$, and how they define $Y_n(x)$.

[4 MARKS]

(ii) Describe what happens to the random variable $T_n(x) = Y_n(x)/n$ as $n \longrightarrow \infty$.

[2 MARKS]

(iii) Suppose that F_X is given by

$$F_X(x) = \frac{x}{1+x} \qquad x > 0$$

and zero otherwise. Evaluate the probability mass function (pmf) of $Y_n(1)$ for $n = 4$, for each real value at which the pmf is non-zero.

[3 MARKS]

(iv) Using the Central Limit Theorem, construct an approximation to the distribution of $T_n(3)$ for large n.

[3 MARKS]