M1S TUTORIAL SHEET : WEEK 10

TRANSFORMATIONS OF RANDOM VARIABLES

For a random variable X with specified probability distribution, it is often necessary to consider the probability distribution of a transformation of X defined by some real-valued function, g say. If X has range $\mathbb{X}$, and g has a domain which includes $\mathbb{X}$, then

$$Y = g(X)$$

is a also a random variable (that is a mapping from Ω to $\mathbb{X}$). We therefore seek to find the probability distribution the range, $\mathbb{Y}$ and mass/density or distribution function of Y.

If X is *discrete*, then Y is also discrete. The range $\mathbb{Y}$ of Y is the image of $\mathbb{X}$ under g, and corresponds to some countable set of values. The mass function of Y, f_Y, is calculated by noting that, for general y,

$$f_Y(y) = \mathrm{P}[\ Y = y\ ] = \mathrm{P}[\ g(X) = y\ ] \equiv \mathrm{P}[X \in A_y]$$

for some set A_y which depends on y. Typically A_y contains a single element of $\mathbb{X}$, but it could contain more than one element, in which case $\mathrm{P}[X \in A_y]$ is computed by summing the probabilities of elements in A_y.

If X is *continuous*, then Y is typically also continuous, and the range $\mathbb{Y}$ of Y is some interval of the real numbers. The probability distribution of Y is calculated via its cumulative distribution function; for general y,

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ g(X) \leq y\ ] \equiv \mathrm{P}[X \in A_y]$$

for some set A_y which depends on y. Finding A_y here is more complicated than in the discrete case, and essentially depends on whether the function g is 1-1 from $\mathbb{X}$ to $\mathbb{Y}$.

If g is 1-1, then the inverse function g^{-1} is also 1-1, and g is either monotonic increasing or monotonic decreasing, in which case

$$F_Y(y) = \mathrm{P}[\ g(X) \leq y\ ] = \begin{cases} \mathrm{P}[\ X \leq g^{-1}(y)\ ] \ = \ F_X(g^{-1}(y)) & g \text{ increasing} \\ \mathrm{P}[\ X \geq g^{-1}(y)\ ] \ = \ 1 - F_X(g^{-1}(y)) & g \text{ decreasing} \end{cases}$$

If g is not 1-1, then $F_Y(y)$ must be calculated by considering the set A_y directly, and noting that

$$F_Y(y) = \mathrm{P}[X \in A_y] = \int_{A_y} f_X(x)\ dx$$

where f_X is the density function of X.

Common examples in which g is not 1-1 include polynomial functions ($X^2, X(1-X)$ etc.) and trigonometric functions.

GENERALIZED EXPECTATIONS

Recall that the **expectation** or **expected value** is one means of describing aspects of a probability distribution. For random variable X with range $\mathbb{X}$, the expectation of X is written $\mathrm{E}_{f_X}[X]$, and defined by

$$\mathrm{E}_{f_X}[X] = \begin{cases} \displaystyle\sum_{x=-\infty}^{\infty} x f_X(x) & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{\infty} x f_X(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

Note that in this definition, terms in the summation, and the integrand, will only be non-zero when $x \in \mathbb{X}$.

More generally, we can also consider the expectation of a function of X, $g(X)$ say; in this case we have that

$$\mathrm{E}_{f_X}[g(X)] = \begin{cases} \displaystyle\sum_{x=-\infty}^{\infty} g(x) f_X(x) & \text{if } X \text{ is discrete} \\ \displaystyle\int_{-\infty}^{\infty} g(x) f_X(x)\, dx & \text{if } X \text{ is continuous} \end{cases}$$

Functions which merit special attention are

(i) the **moments**, where $g(t) = t^k$ for $k = 1, 2, 3, \ldots$ and $\mathrm{E}_{f_X}[g(X)] = \mathrm{E}_{f_X}[X^k]$

(ii) the **central moments**, $g(t) = (t - \mu)^k$ for $k = 1, 2, 3, \ldots$ and

$$\mathrm{E}_{f_X}[g(X)] = \mathrm{E}_{f_X}[(X - \mu)^k],$$

where $\mu = \mathrm{E}_{f_X}[g(X)]$. When $k = 2$, we obtain the **variance** (the second central moment) of the distribution.

The expectation is a measure of *location* of the distribution
The variance is a measure of *scale* or *spread* of the distribution