M1S : EXERCISE SHEET 9 : SOLUTIONS

1. From first principles

(i) $Y_1 = 2X - 1 \Longrightarrow \mathbb{Y}_1 = [-1, 1]$

$$F_{Y_1}(y) = \mathrm{P}[\ Y_1 \leq y\ ] = \mathrm{P}[\ 2X - 1 \leq y\ ] = \mathrm{P}[\ X \leq (1+y)/2\ ] = F_X((1+y)/2)$$

$$\Longrightarrow \quad f_{Y_1}(y) = \frac{1}{2} f_X((1+y)/2) = (1-y)/2 \quad -1 \leq y \leq 1$$

(ii) $Y_2 = 1 - 2X \Longrightarrow \mathbb{Y}_2 = [-1, 1]$

$$F_{Y_2}(y) = \mathrm{P}[\ Y_2 \leq y\ ] = \mathrm{P}[\ 1 - 2X \leq y\ ] = \mathrm{P}[\ X \geq (1-y)/2\ ] = 1 - F_X((1-y)/2)$$

$$\Longrightarrow \quad f_{Y_2}(y) = \frac{1}{2} f_X((1-y)/2) = (1+y)/2 \quad -1 \leq y \leq 1$$

(iii) $Y_3 = X^2 \Longrightarrow \mathbb{Y}_3 = [0, 1]$

$$F_{Y_3}(y) = \mathrm{P}[\ Y_3 \leq y\ ] = \mathrm{P}[\ X^2 \leq y\ ] = \mathrm{P}[\ -\sqrt{y} \leq X \leq \sqrt{y}\ ] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$\Longrightarrow \quad f_{Y_3}(y) = \frac{1}{2\sqrt{y}}\ [\ f_X(\ \sqrt{y}\ ) + f_X(\ -\sqrt{y}\ )\ ] = \frac{1}{2\sqrt{y}}\ f_X(\ \sqrt{y}\ ) = (1-\sqrt{y})/\sqrt{y} \quad 0 \leq y \leq 1$$

(as $f_X(x) = 0$ for $x < 0$)

Could also use the general transformation formula for (i) and (ii), as transformations are 1-1

(i) $g(t) = 2t - 1 \Longleftrightarrow g^{-1}(t) = (1+t)/2$

(ii) $g(t) = 1 - 2t \Longleftrightarrow g^{-1}(t) = (1-t)/2$

then use

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dt} \left\{ g^{-1}(t) \right\}_{t=y} \right|$$

2. Using the general result for expectations of functions with $g(X) = 2\left(1 - e^{-2X}\right)$

$$\begin{aligned}
\mathrm{E}_{f_X}[\ g(X)\ ] &= \int_{-\infty}^{\infty} g(x) f_X(x)\ dx = \int_0^{\infty} 2\left(1 - e^{-2x}\right) \lambda e^{-\lambda x}\ dx \\
&= 2\lambda \int_0^{\infty} \left(e^{-\lambda x} - e^{-(2+\lambda)x}\right)\ dx \\
&= 2\lambda \left[\frac{1}{\lambda} - \frac{1}{2+\lambda}\right] = \frac{4}{2+\lambda} = \frac{1}{2}
\end{aligned}$$

3. $f_X(x) = 1/2, \; -1 \leq x \leq 1$ and zero otherwise. From first principles

(a) $Y = |X| \Longrightarrow \mathbb{Y} = [0,1]$

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ |X| \leq y\ ] = \mathrm{P}[\ -y \leq X \leq y\ ] = F_X(y) - F_X(-y)$$

$$\Longrightarrow \quad f_Y(y) = \ [\ f_X(y) + f_X(-y)\ ] = 1 \quad 0 \leq y \leq 1$$

(b) $Z = X^2 \Longrightarrow \mathbb{Z} = [0,1]$

$$F_Z(z) = \mathrm{P}[\ Z \leq z\ ] = \mathrm{P}[\ X^2 \leq z\ ] = \mathrm{P}[\ -\sqrt{z} \leq X \leq \sqrt{z}\ ] = F_X(\sqrt{z}) - F_X(-\sqrt{z})$$

$$\Longrightarrow \quad f_Z(z) = \frac{1}{2\sqrt{z}}\ [\ f_X(\ \sqrt{z}\ ) + f_X(\ -\sqrt{z}\ )\ ] = \frac{1}{2\sqrt{z}} \quad 0 \leq z \leq 1$$

4. From first principles, if X has cdf F_X, then as X is continuous, F_X is 1-1 and monotone increasing.

(a) $U = F_X(X) \Longrightarrow \mathbb{U} = [0,1]$

$$F_U(u) = \mathrm{P}[\ U \leq u\ ] = \mathrm{P}[\ F_X(X) \leq u\ ] = \mathrm{P}[X \leq F_X^{-1}(u)\ ] = F_X(F_X^{-1}(u)) = u \quad 0 \leq u \leq 1$$

$$\Longrightarrow \quad U \sim Uniform(0,1)$$

(b) $Y = -\log F_X(X) = -\log U \Longrightarrow \mathbb{Y} = \mathbb{R}^+$

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ -\log U \leq y\ ] = \mathrm{P}[\ U \geq e^{-y}\ ] = 1 - F_U(e^{-y}) = 1 - e^{-y} \quad y > 0$$

$$\Longrightarrow \quad Y \sim Exponential(1)$$

5. From first principles, if X has a Weibull distribution, then

$$Y = X^\alpha \Longrightarrow \mathbb{Y} = \mathbb{R}^+$$

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ X^\alpha \leq y\ ] = \mathrm{P}[\ X \leq y^{1/\alpha}\ ] = F_X(y^{1/\alpha}) = 1 - e^{-\beta y} \quad y > 0$$

$$\Longrightarrow \quad Y \sim Exponential(\beta)$$

6.. We have from lectures that if $X \sim Geometric(\theta)$ then

$$M_X(t) = \frac{\theta e^t}{1-(1-\theta)e^t}$$

so that on differentiating twice

$$M_X^{(1)}(t) = \frac{\theta e^t}{\left[1-(1-\theta)e^t\right]^2} \qquad\qquad M_X^{(2)}(t) = \frac{\theta e^t\left[1-(1-\theta)e^t\right]\left[1+(1-\theta)e^t\right]}{\left[1-e^t(1-\theta)e^t\right]^4}$$

so that $M_X^{(1)}(0) = \dfrac{1}{\theta}$ and $M_X^{(2)}(0) = \dfrac{2-\theta}{\theta^2}$, and thus, by the mgf result

$$\mathrm{E}_{f_X}[X] = \frac{1}{\theta} \qquad \mathrm{Var}_{f_X}[X] = \frac{2-\theta}{\theta^2} - \frac{1}{\theta^2} = \frac{1-\theta}{\theta^2}$$

For a direct calculation, can consider $\mathrm{E}_{f_X}[\,X\,] = \sum_{x=1}^{\infty} x f_X(x)$ and $\mathrm{Var}_{f_X}[\,X\,] = \mathrm{E}_{f_X}[\,X^2\,] - \{\mathrm{E}_{f_X}[\,X\,]\}^2$.

However, easier to calculate $\mathrm{E}_{f_X}[\,X\,]$ and $\mathrm{E}_{f_X}[\,X(X-1)\,]$

$$
\begin{aligned}
\mathrm{E}_{f_X}[\,X\,] \quad & = \sum_{x=1}^{\infty} x(1-\theta)^{x-1}\theta = -\theta \sum_{x=0}^{\infty} \frac{d}{d\theta}\{(1-\theta)^x\} = -\theta\,\frac{d}{d\theta}\left\{\sum_{x=0}^{\infty}(1-\theta)^x\right\} \\
& = -\theta\,\frac{d}{d\theta}\left\{\frac{1}{1-(1-\theta)}\right\} = \frac{1}{\theta}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}_{f_X}[\,X(X-1)\,] \quad & = \sum_{x=1}^{\infty} x(x-1)(1-\theta)^{x-1}\theta = \theta(1-\theta)\sum_{x=1}^{\infty} x(x-1)(1-\theta)^{x-2} \\
& = \theta(1-\theta)\sum_{x=0}^{\infty} \frac{d^2}{d\theta^2}\{(1-\theta)^x\} = \theta(1-\theta)\,\frac{d^2}{d\theta^2}\left\{\sum_{x=0}^{\infty}(1-\theta)^x\right\} \\
& = \theta(1-\theta)\,\frac{d^2}{d\theta^2}\left\{\frac{1}{1-(1-\theta)}\right\} = \frac{2(1-\theta)}{\theta^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}_{f_X}[\,X\,] \quad & = \mathrm{E}_{f_X}[\,X^2\,] - \{\mathrm{E}_{f_X}[\,X\,]\}^2 = \mathrm{E}_{f_X}[\,X(X-1)\,] + \mathrm{E}_{f_X}[\,X\,] - \{\mathrm{E}_{f_X}[\,X\,]\}^2 \\
& = \frac{2(1-\theta)}{\theta^2} + \frac{1}{\theta} - \frac{1}{\theta^2} = \frac{(1-\theta)}{\theta^2}
\end{aligned}
$$

Seen that Negative Binomial is sum of independent and identical Geometric random variables; hence

$$
X \sim NegBinomial(n,\theta) \Longrightarrow \mathrm{E}_{f_X}[\,X\,] = \frac{n}{\theta} \qquad \mathrm{Var}_{f_X}[\,X\,] = \frac{n(1-\theta)}{\theta^2}
$$

as it is valid to sum the expectations/variances of the n independent Geometric random variables (to see this result, write out expectation definition in full).

7. If X takes non-negative integer values

$$
\mathrm{E}_{f_X}[X] = \sum_{x=0}^{\infty} x f_X(x) = \sum_{x=0}^{\infty}\sum_{y=1}^{x} f_X(x) = \sum_{y=1}^{\infty}\sum_{x=y}^{\infty} f_X(x) = \sum_{y=1}^{\infty} \mathrm{P}[X \geq y] = \sum_{x=0}^{\infty} \mathrm{P}[X \geq x+1\,] = \sum_{x=0}^{\infty} \mathrm{P}[X > x]
$$

8. By definition $\mathrm{E}_{f_Y}[\,Y\,] = \int_{\mathbb{Y}} y f_Y(y)\,dy$ and result follows by changing variables in this integral from y to x where $y = g(x)$, that is

$$
\mathrm{E}_{f_Y}[\,Y\,] = \int_{\mathbb{Y}} y f_Y(y)\,dy = \int_{\mathbb{X}} g(x) f_Y(g(x))\,\frac{dg(x)}{dx}\,dx \equiv \int_{\mathbb{X}} g(x) f_X(x)\,dx
$$

This result follows immediately from the general transformation result if g is 1-1, and follows for general g by decomposing the range $\mathbb{X}$ into subintervals on which g is monotone, and summing the integrals over these subintervals.

(Thus the two calculation methods are equivalent, and this result justifies the definition for expectations of functions of X.)

9. (i)$X \sim Normal(0,1)$, and thus if Φ and ϕ are the standard normal cdf and pdf respectively. We have immediately that $Y = X^2 \Longrightarrow \mathbb{Y} = \mathbb{R}^+$, and

$$F_Y(y) = \mathrm{P}[\ Y \le y\ ] = \mathrm{P}[\ X^2 \le y\ ] = \mathrm{P}[\ -\sqrt{y} \le X \le \sqrt{y}\ ] = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$$

(ii) By differentiation, we have

$$f_Y(y) = \frac{1}{2\sqrt{y}}\ [\ \phi(\ \sqrt{y}\ ) + \phi(\ -\sqrt{y}\ )\ ] = \left(\frac{1}{2\pi}\right)^{1/2} y^{-1/2}\ e^{-y/2} \qquad 0 \le y \le \infty$$

(iii) By inspection, we have that $Y \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right) \equiv \chi^2_1$

10. Many possible routes; possibly simplest to use mgfs. We have from 9.(iii) that if $X \sim Normal(0,1)$ then

$$Y = X^2 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

and hence

$$M_Y(t) = \int_0^\infty e^{ty}\left(\frac{1}{2\pi}\right)^{1/2} y^{-1/2}\ e^{-y/2}\ dy = \left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty y^{-1/2}\ e^{-y(2-t)/2}\ dy = \left(\frac{1}{2\pi}\right)^{1/2}\frac{(2\pi)^{1/2}}{(2-t)^{1/2}}$$

for $2 > t$, as the integrand is proportional to a $Gamma(1/2, (2-t)/2)$ pdf. Hence

$$M_Y(t) = \left(\frac{1}{2-t}\right)^{1/2}$$

Now, let $Y_1 = X_1^2$ and $Y_2 = X_2^2$, so that Y_1 and Y_2 are independent $Gamma(1/2, 1/2)$ variables. Now we have from a key mgf result

$$V = Y_1 + Y_2 \quad \Longrightarrow \quad M_V(t) = M_{Y_1}(t)M_{Y_2}(t) = \left(\frac{1}{2-t}\right)^{1/2}\left(\frac{1}{2-t}\right)^{1/2} = \left(\frac{1}{2-t}\right)$$

and hence, noting that this is the mgf of a Gamma random variable with parameters 1 and 2, we conclude that

$$V \sim Gamma\left(1, \frac{1}{2}\right) = Gamma\left(\frac{2}{2}, \frac{1}{2}\right) \equiv \chi^2_2$$

By using a joint pdf approach, we could also write

$$F_V(v) = \mathrm{P}\left[\ V \le v\ \right] = \mathrm{P}\left[\ X_1^2 + X_2^2 \le v\ \right] = \int_{A_v}\!\!\int f_{X_1}(x_1)f_{X_2}(x_2)\ dx_1dx_2 \qquad A_v = \left\{(x_1, x_2)\ :\ x_1^2 + x_2^2 \le v\right\}$$

that is, the integral over the region A_v of the joint density function of X_1 and X_2. Now, should reparameterize into polar coordinates in the double integral; let $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Then

$$\begin{aligned} F_V(v) &= \int_{A_v}\!\!\int f_{X_1}(x_1)f_{X_2}(x_2)\ dx_1dx_2 = \int_{A_v}\!\!\int \frac{1}{2\pi}e^{-(x_1^2+x_2^2)/2}\ dx_1dx_2 = \int_{r=0}^{\sqrt{v}}\int_0^{2\pi}\frac{1}{2\pi}e^{-r^2/2}r\ drd\theta \\ &= \int_{r=0}^{\sqrt{v}} re^{-r^2/2}\ dr = 1 - e^{-v/2} \qquad z > 0 \end{aligned}$$

so $V \sim Exponential(1/2) \equiv \chi^2_2$.

Could also use the convolution theorem.