M1S : EXERCISE SHEET 8 : SOLUTIONS

1. (i) By an elementary probability result

$$R_X(x) = \mathrm{P}[X > x] = 1 - \mathrm{P}[X \leq x] = 1 - F_X(x)$$

(ii) By definition, and from (i),

$$h_X(x) = \frac{f_X(x)}{R_X(x)} = \frac{f_X(x)}{1 - F_X(x)} \tag{1}$$

and integrating both sides with respect to x gives

$$H_X(x) = -\log(1 - F_X(x)) \Longleftrightarrow F_X(x) = 1 - \exp\{-H_X(x)\} \Longleftrightarrow R_X(x) = \exp\{-H_X(x)\}$$

as the right hand side of (1) is directly integrable (the numerator is the derivative of the denominator multiplied by -1).

(iii) From (ii),

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x)} \Longleftrightarrow f_X(x) = h_X(x)(1 - F_X(x)) = h_X(x)\exp\{-H_X(x)\}$$

The denominator in the hazard function definition is $R_X(x)$ =P$[X > x]$; hence, recalling the definition of conditional probability, the hazard function is the instantaneous rate of failure at time x, **given** that failure has not occurred before x.

2.Using the definitions and results in 1.,

(i) $f_X(x) = \alpha x^{\alpha-1}\exp\{-x^\alpha\}$.

$$F_X(x) = 1 - \exp\{-x^\alpha\} \qquad R_X(x) = \exp\{-x^\alpha\}$$
$$h_X(x) = \alpha x^{\alpha-1} \qquad H_X(x) = x^\alpha$$

(ii) $f_X(x) = \dfrac{\alpha x^{\alpha-1}}{(1+x^\alpha)^2}$

$$F_X(x) = \frac{x^\alpha}{1+x^\alpha} \qquad R_X(x) = \frac{1}{1+x^\alpha}$$
$$h_X(x) = \frac{\alpha x^{\alpha-1}}{(1+x^\alpha)} \qquad H_X(x) = \log(1+x^\alpha)$$

(iii) $f_X(x) = \dfrac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}$

$$F_X(x) = 1 - \left(\frac{\beta}{\beta+x}\right)^\alpha \qquad R_X(x) = \left(\frac{\beta}{\beta+x}\right)^\alpha$$
$$h_X(x) = \frac{\alpha}{\beta+x} \qquad H_X(x) = \alpha\log(\beta+x)$$

3. In each case we calculate

$$E_{f_X}[X] = \int_{\mathbb{X}} x f_X(x)\, dx$$

and often this involves an integration by parts.

(i)

$$\int_0^\infty x\lambda e^{-\lambda x}\, dx = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}\, dx = 0 + \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda}$$

(ii)

$$\int_{-\infty}^\infty x\lambda e^{-\lambda|x|}\, dx = \int_{-\infty}^0 x\lambda e^{\lambda x}\, dx + \int_0^\infty x\lambda e^{-\lambda x}\, dx = -\frac{1}{\lambda} + \frac{1}{\lambda} = 0$$

Note here that the integral is demonstrably finite, and that the pdf is **symmetric** about zero. Therefore the expectation must be zero; the centre of probability density of a symmetric pdf is always at the point of symmetry.

(iii) Note first that this pdf must integrate to 1 by definition, so immediately we have that for any $k = 1, 2, 3, ...$

$$\int_0^\infty x^k e^{-\lambda x}\, dx = \frac{k!}{\lambda^{k+1}}$$

Now, for the expectation calculation

$$\int_0^\infty x\frac{\lambda^{k+1}}{k!}x^k e^{-\lambda x}\, dx = \frac{\lambda^{k+1}}{k!}\int_0^\infty x^{k+1}e^{-\lambda x}\, dx = \frac{\lambda^{k+1}}{k!}\frac{(k+1)!}{\lambda^{k+2}} = \frac{k+1}{\lambda}$$

(*this trick is used repeatedly in probability calculations*)

(iv) Note first that if parameter $\alpha \leq 1$, the integral **is not convergent** as the integrand (for large x) is approximately proportional to x^α, and hence the expectation is not defined For $\alpha > 1$,

$$\int_0^\infty x\frac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}\, dx = \left[-x\frac{\beta^\alpha}{(\beta+x)^\alpha}\right]_0^\infty + \int_0^\infty \frac{\beta^\alpha}{(\beta+x)^\alpha}\, dx = 0 + \left[-\frac{1}{\alpha-1}\frac{\beta^\alpha}{(\beta+x)^{\alpha-1}}\right]_0^\infty = \frac{\beta}{\alpha-1}$$

(v) The integral **is not convergent** as the integrand (for large x) is approximately proportional to x^{-1}, and hence the expectation is not defined.

(vi) As in (iii), we must have that the pdf integrates to 1 on $[0,1]$. Hence

$$\int_0^1 x^m(1-x)^n\, dx = \frac{m!n!}{(m+n+1)!}$$

so for the expectation calculation, we have

$$\begin{aligned}\int_0^1 x\frac{(m+n+1)!}{m!n!}x^m(1-x)^n\, dx &= \frac{(m+n+1)!}{m!n!}\int_0^1 x^{m+1}(1-x)^n\, dx \\ &= \frac{(m+n+1)!}{m!n!}\frac{(m+1)!n!}{(m+n+2)!} = \frac{m+1}{(m+n+2)}\end{aligned}$$

using the same trick as in (iii).

4. The new generating function for continuous variables is $G_X(t)$ defined by

$$G_X(t) = \int_{\mathbb{X}} t^x f_X(x)\, dx$$

Now, provided this integral is convergent, we differentiate with respect to t under the integral sign. We have

$$\frac{d^r}{dt^r}\left\{t^x f_X(x)\right\} = x(x-1)(x-2)...(x-r+1)t^{x-r}$$

if $r \leq x$, and hence integrating both sides with respect to x we have that

$$\frac{d^r}{dt^r}\left\{G_X(t)\right\} = \int_{\mathbb{X}} x(x-1)(x-2)...(x-r+1)t^{x-r} f_X(x)\, dx$$

Evaluating both sides at $t = 1$ gives

$$\frac{d^r}{dt^r}\left\{G_X(t)\right\}_{t=1} = \int_{\mathbb{X}} x(x-1)(x-2)...(x-r+1) f_X(x)\, dx$$

as required. For $r = 1$, we obtain the expectation.

5. The function $M_X(t)$ for continuous variables is defined by

$$M_X(t) = \int_{\mathbb{X}} e^{tx} f_X(x)\, dx$$

Now, provided this integral is convergent, we differentiate with respect to t under the integral sign. We have

$$\frac{d^r}{dt^r}\left\{e^{tx} f_X(x)\right\} = x^r e^{tx}$$

if $r \leq x$, and hence integrating both sides with respect to x we have that

$$\frac{d^r}{dt^r}\left\{e^{tx} f_X(x)\right\} = \int_{\mathbb{X}} x^r e^{tx} f_X(x)\, dx$$

Evaluating both sides at $t = 0$ gives

$$\frac{d^r}{dt^r}\left\{M_X(t)\right\}_{t=0} = \int_{\mathbb{X}} x^r f_X(x)\, dx = m_r$$

as required.

Now, consider the Taylor expansion of $M_X(t)$ about $t = 0$ given by

$$M_X(t) = M_X(0) + \sum_{r=1} \frac{M_X^{(r)}(0)}{r!} t^r \qquad \text{where } M_X^{(r)}(0) = \frac{d^r}{dt^r}\left\{M_X(t)\right\}_{t=0}$$

We observe that

$$M_X(0) = \int_{\mathbb{X}} e^0 f_X(x)\, dx = \int_{\mathbb{X}} f_X(x)\, dx = 1$$

and also recall from above that $M_X^{(r)}(0) = m_r$, we have that

$$M_X(t) = 1 + \sum_{r=1}^{\infty} \frac{m_r}{r!} t^r$$

as required, which is a generating function for the real coefficients $m_r/r!$, $r = 1, 2, 3, ...$

6. We note first that

$$t^x = e^{x \log t}$$

and therefore, as M_X is defined for all t for which the corresponding integral is convergent, we have

$$G_X(t) = \int_{\mathbb{X}} t^x f_X(x)\, dx = \int_{\mathbb{X}} e^{x \log t} f_X(x)\, dx = M_X(\log t)$$

so that we only calculate M_X for each pdf.

(i) The integral is convergent provided $t < \lambda$, and

$$M_X(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx = \int_0^\infty \lambda e^{-(\lambda - t)x}\, dx = \left[-\frac{\lambda}{\lambda - t} e^{-\lambda x} \right]_0^\infty = \frac{\lambda}{\lambda - t} \qquad \Longrightarrow \qquad G_X(t) = \frac{\lambda}{\lambda - \log t}$$

(ii) Recall that for $k = 1, 2, 3, \ldots$

$$\int_0^\infty x^k e^{-\lambda x}\, dx = \frac{k!}{\lambda^{k+1}}$$

Now, for the expectation calculation

$$M_X(t) = \int_0^\infty e^{tx} \frac{\lambda^{k+1}}{k!} x^k e^{-\lambda x}\, dx = \frac{\lambda^{k+1}}{k!} \int_0^\infty x^k e^{-(\lambda - t)x}\, dx = \frac{\lambda^{k+1}}{k!} \frac{k!}{(\lambda - t)^{k+1}} = \left(\frac{\lambda}{\lambda - t} \right)^{k+1}$$

so that

$$G_X(t) = \left(\frac{\lambda}{\lambda - \log t} \right)^{k+1}$$

(iii) Using the trick mentioned in Q3, we must have

$$\int_{-\infty}^\infty \exp\left\{ -\lambda x^2/2 \right\}\, dx = \left(\frac{2\pi}{\lambda} \right)^{1/2}$$

and hence

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^\infty e^{tx} \left(\frac{\lambda}{2\pi} \right)^{1/2} \exp\left\{ -\lambda x^2/2 \right\}\, dx = \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_{-\infty}^\infty \exp\left\{ -\frac{1}{2} \left[\lambda x^2 - 2tx \right] \right\}\, dx \\
&= \left(\frac{\lambda}{2\pi} \right)^{1/2} \int_{-\infty}^\infty \exp\left\{ -\frac{\lambda}{2} \left[\left(x - \frac{t}{\lambda} \right)^2 + \left(\frac{t}{\lambda} \right)^2 \right] \right\}\, dx \quad \text{(completing the square in } x\text{)} \\
&= \left(\frac{\lambda}{2\pi} \right)^{1/2} \exp\left\{ -\frac{\lambda}{2} \left(\frac{t}{\lambda} \right)^2 \right\} \int_{-\infty}^\infty \exp\left\{ -\frac{\lambda}{2} \left(x - \frac{t}{\lambda} \right)^2 \right\}\, dx \\
&= \left(\frac{\lambda}{2\pi} \right)^{1/2} \exp\left\{ -\frac{t^2}{2\lambda} \right\} \int_{-\infty}^\infty \exp\left\{ -\frac{\lambda}{2} y^2 \right\}\, dy \quad \text{(changing variables to } y = x - t/\lambda\text{)} \\
&= \left(\frac{\lambda}{2\pi} \right)^{1/2} \exp\left\{ -\frac{t^2}{2\lambda} \right\} \left(\frac{2\pi}{\lambda} \right)^{1/2} = \exp\left\{ -\frac{t^2}{2\lambda} \right\}
\end{aligned}$$

so that

$$G_X(t) = \exp\left\{ -\frac{(\log t)^2}{2\lambda} \right\}$$