M1S : EXERCISE SHEET 7 : SOLUTIONS

1. (i) Density function must integrate to 1 over $\mathbb{X} = [0,1]$, so

$$\int_0^1 f_X(x)\ dx = 1 \Longrightarrow \int_0^1 cx^2 + x\ dx = 1 \Longrightarrow \left[c\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = 1 \Longrightarrow c = \frac{3}{2}$$

(ii) Distribution function F_X given for $0 \leq x \leq 1$ by

$$F_X(x) = \int_0^x f_X(t)\ dt = \frac{x^3 + x^2}{2}$$

and $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 1$.

(iii) $\mathrm{P}[\ X < 1/2\ ] = F_X(1/2) = 3/16$.

(iv) From definition of conditional probability

$$\mathrm{P}[\ X > 1/2 \mid X > 1/4\ ] = \frac{\mathrm{P}[\ X > 1/2,\ X > 1/4\ ]}{\mathrm{P}[\ X > 1/4\ ]} = \frac{\mathrm{P}[\ X > 1/2\ ]}{\mathrm{P}[\ X > 1/4\ ]} = \frac{1 - F_X(1/2)}{1 - F_X(1/4)} = \frac{104}{123}$$

(v) $Y \sim Binomial(200, \theta)$ where

$$\theta = \mathrm{P}[\ Y_i = 1\ ] = \mathrm{P}[\ X < 1/6\ ] = F_X(1/6) = \frac{7}{432}$$

Then

$$\mathrm{P}[\ Y \leq 3\ ] = \mathrm{P}[\ Y = 0\ ] + \mathrm{P}[\ Y = 1\ ] + \mathrm{P}[\ Y = 2\ ] + \mathrm{P}[\ Y = 3\ ] = 0.593$$

2. Density function must integrate to 1 over $\mathbb{X} = [-1,1]$, so

$$\int_{-1}^1 f_X(x)\ dx = 1 \Longrightarrow c = \frac{3}{4}$$

Distribution function F_X given for $-1 \leq x \leq 1$ by

$$F_X(x) = \int_{-1}^x f_X(t)\ dt = \left[\frac{3x}{4} - \frac{x^3}{4}\right]_{-1}^1 = \frac{3}{4}\left(x - \frac{x^3}{3}\right) + \frac{2}{3}$$

and $F_X(x) = 0$ for $x < -1$, and $F_X(x) = 1$ for $x > 1$.

3.(i) Density function must integrate to 1 over $\mathbb{X} = [0, \pi/2]$, so

$$\int_0^{\pi/2} f_X(x)\ dx = 1 \Longrightarrow \int_0^{\pi/2} cx(\pi - x)\ dx = 1 \Longrightarrow c = \frac{12}{\pi^3}$$

(ii) $\mathbb{X} = [0, \pi/2] \Longrightarrow \mathbb{Y} = [0, 1/2]$, and distribution function F_Y given by

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ \sin X \leq 2y\ ] = \mathrm{P}[\ X \leq \sin^{-1}(2y)\ ] = F_X(\sin^{-1}(2y))$$

as sin is monotone on $[0, \pi/2]$. Hence by differentiation with respect to y,

$$f_Y(y) = \frac{2}{\sqrt{1 - 4y^2}} f_X(\sin^{-1}(2y)) \qquad 0 \leq y \leq 1/2$$

and zero otherwise.

4. By differentiation, $f_X(x) = 2xe^{-x^2}$, $x > 0$, and zero otherwise

5. Need to consider ranges of integration carefully;

$$F_X(x) = \begin{cases} \displaystyle\int_0^x t\,dt & = x^2/2 \qquad 0 \le x \le 1 \\ \displaystyle\int_0^1 t\,dt + \int_1^x (2-t)\,dt & = 2x - x^2/2 - 1 \quad 1 \le x \le 2 \end{cases}$$

and $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 2$. Hence $\mathrm{P}[\, 0.8 < X < 1.2\,] = F_X(1.2) - F_X(0.8) = 0.36$.

6.(i) $\mathrm{P}[\, T > 3\ |A] = 1 - \mathrm{P}[\, T \le 3\ |A] = 1 - \Phi((3-2)/(3/4)) = 1 - \Phi(4/3)$

(ii) By the Theorem of Total Probability

$$\mathrm{P}[\, T > 3\,] = \mathrm{P}[\, T > 3\ |A]\mathrm{P}(A) + \mathrm{P}[\, T > 3\ |B]\mathrm{P}(B) = (1 - \Phi(4/3)) \times \frac{3}{10} + (1 - \Phi(-4/3)) \times \frac{7}{10}$$

(iii) By Bayes Theorem

$$\mathrm{P}[A|\ T > 3\,] = \frac{\mathrm{P}[\, T > 3\ |A]\mathrm{P}(A)}{\mathrm{P}[\, T > 3\,]}$$

so obtain final answer by evauating $\Phi(4/3)$ via statistical tables, and noting that $\Phi(-4/3) = 1 - \Phi(4/3)$.

7. (i) $X \sim Exponential(1) \Longrightarrow F_X(x) = 1 - e^{-x}$ for $x > 0$,so $F_X(x) = 1/2 \Longrightarrow x = \log 2$.

(ii) $\log X \sim Normal(\mu, \sigma^2)$, so

$$F_X(x) = 1/2 \Longrightarrow \mathrm{P}[\, X \le x\,] = 1/2 \Longrightarrow \mathrm{P}[\, (\log X - \mu)/\sigma \le (\log x - \mu)/\sigma] = 1/2$$

$$\Longrightarrow \Phi((\log x - \mu)/\sigma) = 1/2 \Longrightarrow (\log x - \mu)/\sigma = 0 \Longrightarrow \log x = \mu \Longrightarrow x = e^\mu$$

TUTORIAL

By differentiation

(a) $\dfrac{e^x}{(1+e^x)^2}$ $\qquad x \in \mathbb{R}$

(b) $\alpha\lambda x^{\alpha-1} \exp\left\{-\lambda x^\alpha - e^{-\lambda x^\alpha}\right\}$ $\qquad x \in \mathbb{R}^+$

(c) $\alpha\lambda x^{\alpha-1} e^{-\lambda x^\alpha}$ $\qquad x \in \mathbb{R}^+$

For part (ii), by careful integration

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1/2 \\ 6x - 3x^2 - 2 & 1/2 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$