M1S : EXERCISE SHEET 5 : SOLUTIONS

1. (a) Solution to this occupancy problem given in lectures; allocate $r = 500$ balls to $n = 365$ cells, and count the number of balls in cell 1. Mass function of X is

$$f_X(x) = \binom{500}{x}\left(\frac{1}{365}\right)^x\left(1-\frac{1}{365}\right)^{500-x} \qquad x = 0, 1, 2, ...500$$

(b) If $\lambda = 500/365$, then

$$\begin{aligned}
f_X(x) &= \frac{500!}{x!(500-x)!}\left(\frac{1}{365}\right)^x\left(1-\frac{1}{365}\right)^{500-x} \\
&= \frac{1}{x!}\,\frac{500!}{(500-x)!}\left(\frac{1/365}{1-1/365}\right)^x\left(1-\frac{500}{365.500}\right)^{500} \\
&\approx \frac{1}{x!}\,500^x\left(\frac{1}{365}\right)^x\left(1-\frac{500}{365.500}\right)^{500} = \frac{1}{x!}\,\lambda^x\left(1-\frac{\lambda}{500}\right)^{500} \\
&\approx \frac{\lambda^x e^{-\lambda}}{x!}
\end{aligned}$$

[and hence, in fact, if $X \sim Binomial(n, \theta)$, then if $\theta = \lambda/n$ and $n \longrightarrow \infty$, then X has and approximate Poisson distribution; we will meet these distributions later in the course.]

Numerical verification:

x	0	1	2	3	4	5	6
EXACT	0.2537	0.3484	0.2388	0.1089	0.0372	0.0101	0.0023
APPROXIMATE	0.2541	0.3481	0.2385	0.1089	0.0373	0.0102	0.0023

2. For both variables, range is $\{0, 1, 2\}$, and distribution is given by Hypergeometric formula with $N = 6$, $R = 3$ and $n = 2$. Hence

$$f_X(x) = f_Y(x) = \frac{\binom{3}{x}\binom{3}{2-x}}{\binom{6}{2}} \qquad x = 0, 1, 2$$

and zero otherwise.

3.(a) Range $\mathbb{X} = \{2, 3, 4, 5\}$. Now $f_X(x) = \text{P}[\ X = x\ ] = n_E/n_\Omega$, say, and

$$\begin{aligned}
n_E &= \text{“number of ways of choosing two from five with largest equal to } x\text{”} = x - 1 \\
n_\Omega &= \text{“number of ways of choosing two from five”} = \binom{5}{2} = 10
\end{aligned}$$

so $f_X(x) = \text{P}[\ X = x\ ] = (x-1)/10$.

(b) Range $\mathbb{Y} = \{3, 4, 5, 6, 7, 8, 9\}$ As above, define $f_Y(y) = \text{P}[\ Y = y\ ] = n_E/n_\Omega$, say, and again $n_\Omega = 10$. Enumeration of n_E achieved by considering distinguishable partitions of y into the sum of two integers in the range $\{1, 2, 3, 4, 5\}$. Hence if $y = 3, 4, 8, 9$, $n_E = 1$, but if $y = 5, 6, 7$, $n_E = 2$, so

$$f_Y(y) = \begin{cases} 1/10 & y = 3, 4, 8, 9 \\ 2/10 & y = 5, 6, 7 \end{cases}$$

4. Consider a binary sequence of length $n = 5$ corresponding to the results of the procedures (1=Success, 0=Failure). All such sequences containing x 1s and $n - x$ have probability

$$\theta^x (1-\theta)^{n-x}$$

by the multiplication rule for independent events. Thus the probability that $X = x$ is

$$f_X(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \qquad x = 0, 1, 2, ..., n$$

and zero otherwise, as there are $\binom{n}{x}$ such sequences. Hence $X \sim Binomial(n, \theta)$, so

(i) $\theta = 0.8$, $\text{P}[\ X = 5\ ] = 0.3227$

(ii) $\theta = 0.6$, $\text{P}[\ X = 4\ ] = 0.2592$

(iii) $\theta = 0.3$, $\text{P}[\ X < 2\ ] = \text{P}[\ X = 0\ ] + \text{P}[\ X = 1\ ] = 0.5282$

5. Experiment : sequence of independent and identical binary trials until first success; for $X = x$, need $x - 1$ failures, then a success, and so as all successive tests are independent, we have

$$\begin{aligned} f_X(x) &= (1-\theta)^{x-1}\theta \qquad & x = 1, 2, 3, ... \\ F_X(x) &= 1 - (1-\theta)^x \qquad & x = 1, 2, 3, ... \end{aligned}$$

(so that $X \sim Geometric(\theta)$) and hence

(i) $\theta = 0.25$, $\text{P}[X \leq 3] = F_X(3) = 1 - (1 - 0.25)^3 = 0.5781$

(ii) $\theta = 0.7$, $\text{P}[X > 5] = 1 - \text{P}[X \leq 5] = 1 - F_X(5) = (1 - 0.7)^5 = 0.00243$

6. Experiment : sequence of independent and identical binary trials until third success. For $X = x$, we require that we have $n - 1$ successes in the first $x - 1$ trials, for which we can calculate the probability using the $Binomial(x - 1, \theta)$ formula, and then a success on the xth trial. Hence

$$\text{P}[\ X = x\ ] = \binom{x-1}{n-1} \theta^{n-1} (1-\theta)^{(x-1)-(n-1)} \ \times\ \theta = \binom{x-1}{n-1} \theta^n (1-\theta)^{x-n}$$

for $x = n, n+1, n+2, ...$

Hence we have $X \sim NegBinomial(3, 1/2)$, and hence

$$f_X(x) = \binom{x-1}{n-1} \theta^n (1-\theta)^{x-n} = \binom{x-1}{2}\ (0.5)^3\ (0.5)^{x-3} \qquad x = 3, 4, 5, ...$$

and zero otherwise.