M1S : EXERCISE SHEET 4 : SOLUTIONS

1. (i) Special case of the Binomial Expansion of $(a+b)^n$ with $a=1, b=-1$.

(ii) $(x+1)^{m+n} = (x+1)^m (x+1)^n$, and

$$L.H.S. \quad : \quad (x+1)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k$$

$$R.H.S. \quad : \quad (x+1)^m (x+1)^n = \left\{ \sum_{i=0}^{m} \binom{m}{i} x^i \right\} \left\{ \sum_{j=0}^{n} \binom{n}{j} x^j \right\}$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} x^{i+j}$$

$$= \sum_{k=0}^{m+n} \left\{ \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} \right\} x^k$$

putting $i+j=k$, and re-arranging summations. Result follows by equating left and right hand sides and comparing coefficients of x_k.

2. Hypergeometric Probabilities with $N=49$, $R=6$ (winning balls), and $n=6$ (drawn balls).

(a) $r=6$ Probability $= \dfrac{\binom{6}{6}\binom{43}{0}}{\binom{49}{6}} = \dfrac{1}{13983816} \approx 7.15 \times 10^{-8}$

(b) $r=3$ Probability $= \dfrac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} \approx 0.0177$

(c) $r=0$ P(No winning numbers) $= \dfrac{\binom{n}{r}\binom{N-n}{R-r}}{\binom{N}{R}} = \dfrac{\binom{43}{6}}{\binom{49}{6}}$

(d) P(No winning numbers for x weeks) $= \left\{ \dfrac{\binom{43}{6}}{\binom{49}{6}} \right\}^x$ by independence

3. Hypergeometric probabilities with $N = 80$, $R = 20$, $n = 5, 10, 15$, $r = 0$, for which probabilities are

$$\text{(i)}\ \frac{\binom{5}{0}\binom{75}{20}}{\binom{80}{20}} = \frac{\binom{20}{0}\binom{60}{5}}{\binom{80}{5}} \quad \text{(ii)}\ \frac{\binom{10}{0}\binom{70}{20}}{\binom{80}{20}} = \frac{\binom{20}{0}\binom{60}{10}}{\binom{80}{10}} \quad \text{(iii)}\ \frac{\binom{15}{0}\binom{65}{20}}{\binom{80}{20}} = \frac{\binom{20}{0}\binom{60}{15}}{\binom{80}{15}}$$

General formula is just the Hypergeometric formula.

4. This is a Hypergeometric problem with $N = 200$, $R = 120$, $n = 5$ and $r = 5$, thus the probability required is

$$\frac{\binom{5}{5}\binom{200-5}{120-5}}{\binom{200}{120}} = \frac{120 \times 119 \times 118 \times 117 \times 116}{200 \times 199 \times 198 \times 197 \times 196} = 0.075$$

which may also be calculated using a conditional probability argument.

Thus the probability that an all-male committee is selected, if all selections of five from two hundred are equally likely, is 0.075. This is small, but typically we consider a probability of 0.05 as providing sufficient evidence against a hypothesized model, so perhaps this is not conclusive evidence of sex-bias.

5. (a) Choose 13 cards from 52 $\Longrightarrow n = \binom{52}{13}$

(b) Choose 13 cards from 36 $\Longrightarrow n_E = \binom{36}{13} \Longrightarrow \mathrm{P}(E) = \dfrac{n_E}{n_\Omega} \approx 0.003$

6. Total number of allocations of birthdays to months is 12^{30}. For n_E, select 6 months from 12 as the "two-birthday" months, and then *partition* the 30 birthdays into 12 subgroups, six which contain 2 birthdays, and six which contain three. Hence, using the MULTINOMIAL FORMULA for partitions,

$$n_E = \binom{12}{6}\frac{30!}{(2!)^6(3!)^6} \qquad \Longrightarrow \qquad \mathrm{P}(E) = \binom{12}{6}\frac{30!}{12^{30}(2!)^6(3!)^6}$$

7. Hypergeometric: $2n$ females (TYPE I) and $2n$ males (TYPE II), sample of size $2n$ without replacement from population of size $4n$. Hence probability of even split in each group is

$$\frac{\binom{2n}{n}\binom{2n}{n}}{\binom{4n}{2n}}$$

8. Multistage Hypergeometric: select the first hand with n then the second hand with m all without replacement. Hence by multiplication theorem, probability is

$$\frac{\binom{13}{n}\binom{39}{13-n}}{\binom{52}{13}} \times \frac{\binom{13-n}{m}\binom{26+n}{13-m}}{\binom{39}{13}}$$

By extension, need to select third hand in similar fashion, but once first three hands are selected, final hand is fixed. Hence probability is

$$\frac{\binom{13}{r_1}\binom{39}{13-r_1}}{\binom{52}{13}} \times \frac{\binom{13-r_1}{r_2}\binom{26+r_1}{13-r_2}}{\binom{39}{13}} \times \frac{\binom{13-r_1-r_2}{r_3}\binom{13+r_1+r_2}{13-r_3}}{\binom{26}{13}}$$

9. Total number of seating arrangements is $n!$. Of these, you and friend sit together in

$$(n-1) \times 2! \times (n-2)!$$

arrangements (choose a pair of adjacent seats for you and friend from $n-1$ pairs available, arrange yourselves in one of 2! ways, and then arrange remaining $n-2$ people in remaining seats). Hence probability is

$$\frac{(n-1) \times 2! \times (n-2)!}{n!} = \frac{2}{n}$$

10. Total number of possible allocations is $n \times n \times ... \times n = n^n$ (MULTIPLICATION RULE). For n_E, consider selecting two different boxes from n to be the empty box and the box containing two balls; after this selection has been made, can arrange the balls in the boxes in $n!$ ways; hence $n_E = \binom{n}{2} \times n!$, and

$$\mathrm{P}(E) = \frac{\binom{n}{2} \times n!}{n^n}$$

11. (i) Total number of arrangements of $2n$ objects is $(2n)!$. For n_E, there are n *pairs* of *matched* short and long parts, which can be arranged in $n!$ ways in the sequence; each pair can be arranged in 2 ways. Hence $n_E = 2^n n!$, and hence

$$\mathrm{P}(E) = \frac{2^n n!}{(2n)!}$$

(ii) Here, for n_E, consider arranging a sequence of n short parts and a parallel sequence of n long parts to match with them; there are $n!$ ways of arranging the short parts and $n!$ ways of arranging the long parts. Once the parallel sequences have been arranged individually and paired, each pair can be arranged in 2 ways (long/short or short/long). Hence $n_E = 2^n n! n!$, and

$$\mathrm{P}(E) = \frac{2^n n! n!}{(2n)!} = \frac{2^n}{\binom{2n}{n}}$$

TUTORIAL

There are two alternate forms for the probability that results from a hypergeometric sampling experiment; if a finite population of size N comprises R Type I objects and $N-R$ Type II objects, and a sample of size $n \leq N$ is obtained from the population, then the probability that the sample contains r Type I objects and $n-r$ Type II objects is either expressed

$$\frac{\binom{n}{r}\binom{N-n}{R-r}}{\binom{N}{R}}$$

or

$$\frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}}$$

First, to check the equality of Hypergeometric Formulae, just need to write out factorial terms in full. Recall that

Method A: Consider choosing items in the population to label Type I and Type II, broken down by whether they are IN the sample or OUT of the sample. The denominator is the total number of ways of choosing R from N items to label Type I. The numerator is the number of ways of choosing r from n items IN the sample to be Type I items, multiplied by the number of ways of choosing the $R-r$ from $N-R$ items OUT of the sample to be Type II items.

Method B: Consider items to be IN the sample, broken down by Type. The denominator is the total number of ways of choosing n items from N to be IN the sample. The numerator is the number of ways of choosing r Type I items from R to be IN the sample, multiplied by the number of ways of choosing the $n-r$ Type II items from $N-R$ to be IN the sample.

For other specific problems,

(a) Birthday problem will be studied in lectures;

$$\mathrm{P}(E) = \frac{n_E}{n_\Omega} = \frac{365 \times 364 \times \ldots \times (365-N+1)}{365 \times 365 \times \ldots \times 365} = \frac{365!}{(365-N)!}\frac{1}{365^N} = \frac{(365)_N}{365^N}$$

(b) Occupancy problem; allocate six "balls" to six "cells" with no cells empty, that is, with one ball in each cell.

$$\mathrm{P}(E) = \frac{6!}{6^6}$$

(c) Poker hands; denominator is always $n_\Omega = \binom{52}{5}$.

For the numerator, using generic notation let x, y be the "scoring" cards and a, b, c be the remaining ones. In calculating the probablities, list the scoring denominations in descending order, then for each in turn multiply the number of remaining ways of choosing the denomination by the number of ways of choosing the suits for that denomination. For example, for Full House ($xxxyy$), you get

$$\binom{13}{1}\binom{4}{3} \times \binom{12}{1}\binom{4}{2}$$

In the case of ties in the list of scoring denominations (i.e. two x and two y in the hand), remember that you have to choose two **different** denominations so that for Two Pairs ($xxyya$) the number of ways of choosing the scoring cards is

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}$$

Hand		n_E	Prob.
ONE PAIR	*xxabc*	$\binom{13}{1}\binom{4}{2} \times \binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$	0.42
TWO PAIRS	*xxyya*	$\binom{13}{2}\binom{4}{2}\binom{4}{2} \times \binom{11}{1}\binom{4}{1}$	0.048
THREE OF A KIND	*xxxab*	$\binom{13}{1}\binom{4}{3} \times \binom{12}{2}\binom{4}{1}\binom{4}{1}$	0.021
STRAIGHT	Run of five cards	$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}$	0.0039
FLUSH	Five cards in same suit	$\binom{13}{5}\binom{4}{1} - \binom{10}{1}\binom{4}{1}$	0.0020
FULL HOUSE	*xxxyy*	$\binom{13}{1}\binom{4}{3} \times \binom{12}{1}\binom{4}{2}$	0.0014
FOUR OF A KIND	*xxxxa*	$\binom{13}{1}\binom{4}{4} \times \binom{12}{1}\binom{4}{1}$	0.00024
STRAIGHT FLUSH	Run of five cards in same suit	$10\binom{4}{1} - \binom{4}{1}$	0.000014
ROYAL FLUSH	AKQJ10 in any suit	$1\binom{4}{1}$	0.0000015

For STRAIGHT and FLUSH, must remember to subtract the STRAIGHT FLUSHes (which are counted separately); similarly, for STRAIGHT FLUSH, must remember to subtract ROYAL FLUSHes.