M1S : EXERCISE SHEET 3 : SOLUTIONS

1. General proofs given in lectures; for events $E_1, E_2, F \subseteq \Omega$ with $\mathrm{P}(F) > 0$;

$$\text{(I)} \quad \mathrm{P}(E_1|F) = \frac{\mathrm{P}(E_1 \cap F)}{\mathrm{P}(F)} \geq 0, \text{and } E_1 \cap F \subseteq F \Longrightarrow \mathrm{P}(E_1|F) \leq 1.$$

$$\text{(II)} \quad \mathrm{P}(\Omega|F) = \frac{\mathrm{P}(\Omega \cap F)}{\mathrm{P}(F)} = \frac{\mathrm{P}(F)}{\mathrm{P}(F)} = 1$$

$$\text{(III)} \quad \mathrm{P}(E_1 \cup E_2|F) = \frac{\mathrm{P}((E_1 \cup E_2) \cap F)}{\mathrm{P}(F)} = \frac{\mathrm{P}(E_1 \cap F)}{\mathrm{P}(F)} + \frac{\mathrm{P}(E_1 \cap F)}{\mathrm{P}(F)} = \mathrm{P}(E_1|F) + \mathrm{P}(E_2|F)$$

as $(E_1 \cup E_2) \cap F = (E_1 \cap F) \cup (E_2 \cap F)$ which are disjoint events. Verification in the relative frequency interpretation follows exactly the proof given in lectures for the classical interpretation; let n_{TOT} be the total number of repeats, let n_{E_1}, $n_{E_1 \cap F}$ etc. be the numbers of times that the corresponding event occurs. Then let $n_{TOT} \longrightarrow \infty$.

2.

$$\begin{aligned}
\text{(a)} \quad \mathrm{P}(E' \cap F) &= \mathrm{P}(F) - \mathrm{P}(E \cap F) = \mathrm{P}(F) - \mathrm{P}(E)\mathrm{P}(F) = (1 - \mathrm{P}(E))\mathrm{P}(F) = \mathrm{P}(E')\mathrm{P}(F) \\
\mathrm{P}(E' \cap F') &= 1 - \mathrm{P}(E \cup F) = 1 - \mathrm{P}(E) - \mathrm{P}(F) + \mathrm{P}(E \cap F) \\
&= 1 - \mathrm{P}(E) - \mathrm{P}(F) + \mathrm{P}(E)\mathrm{P}(F) = (1 - \mathrm{P}(E))(1 - \mathrm{P}(F)) = \mathrm{P}(E')\mathrm{P}(F')
\end{aligned}$$

$$\text{(b)} \quad \mathrm{P}(E \cap F) = 0 \Longleftrightarrow \mathrm{P}(E)\mathrm{P}(F) = 0 \Longleftrightarrow \text{at least one of } \mathrm{P}(E), \mathrm{P}(F) = 0.$$

3.

$$\begin{aligned}
\text{(a)} \quad \mathrm{P}(A) &= \mathrm{P}(A_1 \cup A_2) = \mathrm{P}(A_1) + \mathrm{P}(A_2) - \mathrm{P}(A_1 \cap A_2) \\
&= 0.95 + 0.9 - 0.95 \times 0.9 = 0.995 \\
\text{(b)} \quad \mathrm{P}(B) &= \mathrm{P}(B_1 \cap B_2 \cap B_3') + \mathrm{P}(B_1 \cap B_2' \cap B_3) + \mathrm{P}(B_1' \cap B_2 \cap B_3) + \mathrm{P}(B_1 \cap B_2 \cap B_3) \\
&= (0.8 \times 0.8 \times 0.2) + (0.8 \times 0.2 \times 0.8) + (0.2 \times 0.8 \times 0.8) + (0.8 \times 0.8 \times 0.8) \\
&= 0.896 \\
\text{(c)} \quad \mathrm{P(S)} &= \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C)\mathrm{P}(D)\mathrm{P}(E) = 0.995 \times 0.896 \times 0.95^3 = 0.764
\end{aligned}$$

4. Without loss of generality, let events A, B, C correspond to the prize being behind the selected, opened, and remaining door respectively, and let H_B denote the event that the host opens door B. Want to compare $\mathrm{P}(A|H_B)$ (STICK) with $\mathrm{P}(C|H_B)$ (SWITCH). Now $\mathrm{P}(A) = \mathrm{P}(B) = \mathrm{P}(C) = 1/3$, and we are given that $\mathrm{P}(H_B|A) = 1/2$, $\mathrm{P}(H_B|B) = 0$ and $\mathrm{P}(H_B|C) = 1$. Then the general version of Bayes theorem gives

$$\begin{aligned}
\mathrm{P}(A|H_B) &= \frac{\mathrm{P}(H_B|A)\mathrm{P}(A)}{\mathrm{P}(H_B)} = \frac{\mathrm{P}(H_B|A)\mathrm{P}(A)}{\mathrm{P}(H_B|A)\mathrm{P}(A) + \mathrm{P}(H_B|B)\mathrm{P}(B) + \mathrm{P}(H_B|C)\mathrm{P}(C)} \\
&= \frac{\frac{1}{2}\frac{1}{3}}{\frac{1}{2}\frac{1}{3} + 0\frac{1}{3} + 1\frac{1}{3}} = \frac{1}{3}
\end{aligned}$$

so $\mathrm{P}(A|H_B) = 1/3$, and similarly $\mathrm{P}(C|H_B) = 2/3$. so it is advantageous to SWITCH.

5. (i) Given $\mathrm{P}(A) = \mathrm{P}(B) = \mathrm{P}(C) = 1/3$, and $\mathrm{P}(G_{AB}|A) = 1/2$, $\mathrm{P}(G_{AB}|B) = 0$ and $\mathrm{P}(G_{AB}|C) = 1$, and hence by Bayes theorem using an identical calculation to above, we have $\mathrm{P}(A|G_{AB}) = 1/3$ and hence governor is correct.

(ii) Now $\mathrm{P}(G_{WB}|A) = 1/2$, $\mathrm{P}(G_{WB}|B) = 0$ but $\mathrm{P}(G_{WB}|C) = 1/2$, so by Bayes theorem $\mathrm{P}(C|G_{WB}) = 1/2$, and hence C is right to feel happier.

6. Given $\mathrm{P}(G) = p$, $\mathrm{P}(A|G) = 1$, $\mathrm{P}(A|G') = \pi$. Then

$$\mathrm{P}(G|A) = \frac{\mathrm{P}(A|G)\mathrm{P}(G)}{\mathrm{P}(A|G)\mathrm{P}(G) + \mathrm{P}(A|G')\mathrm{P}(G')} = \frac{1 \times p}{1 \times p + \pi \times (1-p)} \quad \Longrightarrow \quad \frac{\mathrm{P}(G|A)}{\mathrm{P}(G'|A)} = \frac{p}{\pi \times (1-p)} = \frac{\mathrm{P}(G)}{\pi \mathrm{P}(G')}$$

7. Let $T \equiv$ "Test positive", $S \equiv$ "Sufferer". Then $\mathrm{P}(T|S) = 0.95$, $\mathrm{P}(T|S') = 0.10$, $\mathrm{P}(S) = 0.005$. Hence

(a) $\mathrm{P}(T) \; = \; = \mathrm{P}(T|S)\mathrm{P}(S) + \mathrm{P}(T|S')\mathrm{P}(S')(0.95 \times 0.005) + (0.1 \times 0.995) = 0.10425$

(b) $$\mathrm{P}(S|T) = \frac{\mathrm{P}(T|S)\mathrm{P}(S)}{\mathrm{P}(T|S)\mathrm{P}(S) + \mathrm{P}(T|S')\mathrm{P}(S')} = \frac{0.95 \times 0.005}{(0.95 \times 0.005) + (0.1 \times 0.995)} = 0.0455$$

(c) $$\mathrm{P}(S'|T') = \frac{\mathrm{P}(T'|S')\mathrm{P}(S')}{\mathrm{P}(T')} = \frac{0.9 \times 0.995}{1 - 0.10425} = 0.9997$$

(d) $\mathrm{P}(M) = \mathrm{P}(T \cap S') + \mathrm{P}(T' \cap S) = \mathrm{P}(T|S')\mathrm{P}(S') + \mathrm{P}(T'|S)\mathrm{P}(S) = 0.09975$

8. Let $T_1 \equiv$ "first test positive", $T_2 \equiv$ "second test positive", $C \equiv$ "drugs present in sample". Then given that

$$\mathrm{P}(T_1|C) = \mathrm{P}(T_2|C) = 0.995 \qquad \mathrm{P}(T_1'|C') = \mathrm{P}(T_2'|C') = 0.98.$$

(a) By the Theorem of Total Probability

$$\mathrm{P}(T_1) = \mathrm{P}(T_1|C)\mathrm{P}(C) + \mathrm{P}(T_1|C')\mathrm{P}(C') = 0.995 \times 0.001 + (1 - 0.98) \times 0.999 = 0.021.$$

(b) By Bayes Theorem

$$\mathrm{P}(C|T_1) = \frac{\mathrm{P}(T_1|C)\mathrm{P}(C)}{\mathrm{P}(T_1|C)\mathrm{P}(C) + \mathrm{P}(T_1|C')\mathrm{P}(C')} = \frac{0.995 \times 0.001}{0.995 \times 0.001 + (1 - 0.98) \times 0.999} = 0.047.$$

(c) By the Theorem of Total Probability and conditional independence

$$\begin{aligned}\mathrm{P}(T_1 \cap T_2) &= \mathrm{P}(T_1 \cap T_2|C)\mathrm{P}(C) + \mathrm{P}(T_1 \cap T_2|C')\mathrm{P}(C') = \mathrm{P}(T_1|C)\mathrm{P}(T_2|C)\mathrm{P}(C) + \mathrm{P}(T_1|C')\mathrm{P}(T_2|C')\mathrm{P}(C') \\ &= 0.995^2 \times 0.001 + (1 - 0.98)^2 \times 0.999 = 0.014.\end{aligned}$$

(d) By Bayes Theorem

$$\begin{aligned}\mathrm{P}(C|T_1 \cap T_2) &= \frac{\mathrm{P}(T_1 \cap T_2|C)\mathrm{P}(C)}{\mathrm{P}(T_1 \cap T_2|C)\mathrm{P}(C) + \mathrm{P}(T_1 \cap T_2|C')\mathrm{P}(C')} = \frac{\mathrm{P}(T_1|C)\mathrm{P}(T_2|C)\mathrm{P}(C)}{\mathrm{P}(T_1|C)\mathrm{P}(T_2|C)\mathrm{P}(C) + \mathrm{P}(T_1|C')\mathrm{P}(T_2|C')\mathrm{P}(C')} \\ &= \frac{0.995^2 \times 0.001}{0.995^2 \times 0.001 + (1 - 0.98)^2 \times 0.999} = 0.712\end{aligned}$$

TUTORIAL SHEET WEEK 4 : SOLUTIONS

(a) Can represent each event as a disjoint union of a subset of the events corresponding to the cells in the table, that is, the events

	Event	Entry
1.	$F \cap D \cap M_1$	20
2.	$F \cap D' \cap M_1$	16
3.	$F \cap D \cap M_2$	30
4.	$F \cap D' \cap M_2$	20
5.	$F \cap D \cap M_3$	15
6.	$F \cap D' \cap M_3$	10
7.	$F' \cap D \cap M_1$	100
8.	$F' \cap D' \cap M_1$	64
9.	$F' \cap D \cap M_2$	120
10.	$F' \cap D' \cap M_2$	30
11.	$F' \cap D \cap M_3$	60
12.	$F' \cap D' \cap M_3$	15

- can just find a partition for the event of interest, then sum the probabilities using Axiom (III). For conditional probabilities, can use the conditional probability definition and proceed using partitions and Axiom (III).

(a) (i) $$\mathrm{P}(F) = \frac{20+16+30+20+15+16}{500} = \frac{111}{500}$$

(ii) $$\mathrm{P}(M_1) = \frac{20+16+100+64}{500} = \frac{200}{500}$$

(b) (i) $$\mathrm{P}(D \mid F) = \frac{\mathrm{P}(D \cap F)}{\mathrm{P}(F)} = \frac{20+30+15}{20+16+30+20+15+16} = \frac{65}{111}$$

(ii) $$\mathrm{P}(M_1 \mid F) = \frac{\mathrm{P}(M_1 \cap F)}{\mathrm{P}(F)} = \frac{20+16}{20+16+30+20+15+16} = \frac{36}{111}$$

(iii) $$\mathrm{P}(D \cap M_1 \mid F) = \frac{\mathrm{P}(D \cap M_1 \cap F)}{\mathrm{P}(F)} = \frac{20}{20+16+30+20+15+16} = \frac{20}{111}$$

(c) (i) $$\mathrm{P}(F \mid M_1) = \frac{\mathrm{P}(F \cap M_1)}{\mathrm{P}(M_1)} = \frac{20+16}{20+16+100+64} = \frac{36}{200}$$

$$\mathrm{P}(F \mid M_2) = \frac{\mathrm{P}(F \cap M_2)}{\mathrm{P}(M_2)} = \frac{30+20}{30+20+120+30} = \frac{50}{200}$$

$$\mathrm{P}(F \mid M_3) = \frac{\mathrm{P}(F \cap M_3)}{\mathrm{P}(M_3)} = \frac{15+10}{15+10+60+15} = \frac{35}{100}$$

(ii) $$\mathrm{P}(F \mid D) = \frac{\mathrm{P}(F \cap D)}{\mathrm{P}(D)} = \frac{20+30+15}{20+30+15+100+120+60} = \frac{65}{345}$$

$$\mathrm{P}(F \mid D') = \frac{\mathrm{P}(F \cap D')}{\mathrm{P}(D')} = \frac{16+20+10}{16+20+10+64+30+15} = \frac{46}{155}$$

(iii) $$\mathrm{P}(F \mid M_1 \cap D) = \frac{\mathrm{P}(F \cap M_1 \cap D)}{\mathrm{P}(M_1 \cap D)} = \frac{20}{20+100} = \frac{20}{120}$$

$$\mathrm{P}(F \mid M_2 \cap D) = \frac{\mathrm{P}(F \cap M_2 \cap D)}{\mathrm{P}(M_2 \cap D)} = \frac{30}{30+120} = \frac{30}{150}$$

$$\mathrm{P}(F \mid M_3 \cap D) = \frac{\mathrm{P}(F \cap M_3 \cap D)}{\mathrm{P}(M_3 \cap D)} = \frac{15}{15+60} = \frac{15}{75}$$

(iv) $$\mathrm{P}(F \mid M_1 \cap D') = \frac{\mathrm{P}(F \cap M_1 \cap D')}{\mathrm{P}(M_1 \cap D')} = \frac{16}{16+64} = \frac{16}{80}$$

$$\mathrm{P}(F \mid M_2 \cap D') = \frac{\mathrm{P}(F \cap M_2 \cap D')}{\mathrm{P}(M_2 \cap D')} = \frac{20}{20+30} = \frac{20}{50}$$

$$\mathrm{P}(F \mid M_3 \cap D') = \frac{\mathrm{P}(F \cap M_3 \cap D')}{\mathrm{P}(M_3 \cap D')} = \frac{10}{10+15} = \frac{10}{25}$$

These results confirm that events F, M_1, M_2, M_3 and D are not mutually independent.