M1S : EXERCISE SHEET 2 : SOLUTIONS

1.(a) Partition according to disease status : $(S \cap T') \cup (S' \cap T)$

(b) Partition first into correctly classified sufferers/non-sufferers

(i)	Sufferers	:	$(D \cap (X \cup T) \cap S)$
(i)	Non-sufferers	:	$(((D \cap (X \cup T)') \cup D') \cap S')$

(i) follows as sufferers must have the correct doctor's diagnosis, and at least one the tests indicating the disease. (ii) follows as non-sufferers can either receive the incorrect doctor's diagnosis (and be declared as sufferers), but be classified correctly as they recieve (correct) negative X-ray and test results, or merely receive a (correct) negative test diagnosis from the doctor.

2. Let G="exactly one occurs". Then $G = (E \cap F') \cup (E' \cap F)$, and by Axiom (III) $\text{P}(G) = \text{P}(E \cap F') + \text{P}(E' \cap F)$. Also,

$$E = (E \cap F) \cup (E \cap F') \implies \text{P}(E) = \text{P}(E \cap F) + \text{P}(E \cap F')$$
$$F = (E \cap F) \cup (E' \cap F) \implies \text{P}(F) = \text{P}(E \cap F) + \text{P}(E' \cap F)$$

so therefore

$$\text{P}(G) = \text{P}(E) + \text{P}(F) - 2\text{P}(E \cap F).$$

3. (a) FALSE : (e.g. $E_1 = \emptyset$, $E_2 = \Omega$).

(b) FALSE : (e.g. coin toss, let $E_1 = E_2 = \{H\}$).

(c) TRUE : $E_1 \subseteq E_2 \implies \text{P}(E_1) \leq \text{P}(E_2) = 1 - \text{P}(E_2') = 1 - \text{P}(E_1) \implies 2\text{P}(E_1) \leq 1$.

(d) TRUE : $\text{P}(E_1 \cup E_2) = \text{P}(E_1) + \text{P}(E_1) - \text{P}(E_1 \cap E_2) \geq (1 - x_1) + (1 - x_2) - 1 = 1 - x_1 - x_2$.

4. For general events E and F,
(a) $F \equiv (E \cap F) \cup (E' \cap F)$, so by Axiom (III)

$$\text{P}(F) = \text{P}(E \cap F) + \text{P}(E' \cap F) \implies \text{P}(E' \cap F) = \text{P}(F) - \text{P}(E \cap F)$$

(b) $E \cup F \equiv E \cup (E' \cap F)$, so by Axiom (III)

$$\text{P}(E \cup F) = \text{P}(E) + \text{P}(E' \cap F) = \text{P}(E) + \text{P}(F) - \text{P}(E \cap F)$$

(c) $E \subseteq F \implies F \equiv E \cup (E' \cap F)$, so by Axiom (III), as $\text{P}(E' \cap F) \geq 0$,

$$\text{P}(F) = \text{P}(E) + \text{P}(E' \cap F) \geq \text{P}(E)$$

(d) *Bonferroni Inequality*: From (ii), as $\text{P}(E \cup F) \leq 1$,

$$\text{P}(E \cap F) = \text{P}(E) + \text{P}(F) - \text{P}(E \cup F) \geq \text{P}(E) + \text{P}(F) - 1$$

5. (a) $E' \cup F' = (E \cap F)' \implies \text{P}(E' \cup F') = 1 - \text{P}(E_1 \cap E_2) = 1 - z$.

(b) $F = (E \cap F) \cup (E' \cap F)$, so $\text{P}(F) = \text{P}(E \cap F) + \text{P}(E' \cap F)$, so $\text{P}(E' \cap F) = y - z$.

(c) $E' \cup F = E' \cup (E \cap F) \implies \text{P}(E' \cup F) = \text{P}(E') + \text{P}(E \cap F) = 1 - x + z$.

(d) $E' \cap F' = (E \cup F)' \implies \text{P}(E' \cap F') = 1 - \text{P}(E \cup F) = 1 - x - y + z$.

6. $E \cup E' = \Omega$, so by Axioms (II) and (III), $\text{P}(E) + \text{P}(E') = \text{P}(\Omega) = 1$. Hence $\text{P}(E') = 1 - \text{P}(E)$. But, by Axiom (I)*, $\text{P}(E') \geq 0$, hence $\text{P}(E) \leq 1$.

7. (a) True for $n = 2$ as $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \leq \mathrm{P}(A) + \mathrm{P}(B)$. So assume true for $n = k$; then

$$\mathrm{P}(E_1 \cup ... \cup E_k \cup E_{k+1}) \leq \mathrm{P}(E_1 \cup ... \cup E_k) + \mathrm{P}(E_{k+1}) = \sum_{i=1}^{k+1} \mathrm{P}(E_i)$$

and hence true for $n = k + 1$.

(b) As for (a) except

$$1\mathrm{P}(E_1 \cup ... \cup E_k \cup E_{k+1}) = \mathrm{P}(E_1 \cup ... \cup E_k) + \mathrm{P}(E_{k+1}) - \mathrm{P}\left((E_1 \cup ... \cup E_k) \cap E_{k+1}\right). \tag{1}$$

Result follows by substituting required form for $\mathrm{P}(E_1 \cup ... \cup E_k)$, and re-writing

$$\mathrm{P}\left((E_1 \cup ... \cup E_k) \cap E_{k+1}\right) = \mathrm{P}\left((E_1 \cap E_{k+1}) \cup ... \cup (E_k \cap E_{k+1})\right)$$

(using distributivity) which is the union of k events, and hence can use the inductive hypothesis to re-write this final expression in the required form. Specifically, let $F_i = E_i \cap E_{k+1}$; then we have

$$\begin{aligned} \mathrm{P}(F_1 \cup F_2 \cup ... \cup F_k) &= \sum_i \mathrm{P}(F_i) - \sum_i \sum_j \mathrm{P}(F_i \cap F_j) + ... (-1)^{k-1} \mathrm{P}(F_1 \cap F_2 \cap ... \cap F_k) \\ &= \sum_i \mathrm{P}(E_i \cap E_{k+1}) - \sum_i \sum_j \mathrm{P}(E_i \cap E_j \cap E_{k+1}) + ... \\ &\qquad (-1)^{k-1} \mathrm{P}(E_1 \cap E_2 \cap ... \cap E_k \cap E_{k+1}) \end{aligned}$$

as, for example,

$$F_i \cap F_j = (E_i \cap E_{k+1}) \cap (E_j \cap E_{k+1}) = E_i \cap E_j \cap E_{k+1}$$

Hence, using the inductive hypothesis to express

$$\mathrm{P}(E_1 \cup ... \cup E_k)$$

in (1) and adding in the final term, we complete the proof.

TUTORIAL SHEET WEEK 3 : SOLUTIONS

1. Try to verify Axioms (I), (II) and (III) within each framework.

RELATIVE FREQUENCY: $\mathrm{P}(E) = \dfrac{n_E}{n}$ as $n \longrightarrow \infty$.

(I) OK, as n_E/n always lies in $[0,1]$.

(II) OK, as $n_\Omega = n$, so $\mathrm{P}(\Omega) = n_\Omega/n = 1$.

(III) OK, as $n_{E\cup F} = (n_E + n_F)$ if E and F are disjoint, so $\mathrm{P}(E\cup F) = (n_E+n_F)/n = n_E/n + n_F/n = \mathrm{P}(E)+\mathrm{P}(F)$ as $n \longrightarrow \infty$.

CLASSICAL: Identical to RF case, but for finite n.

SUBJECTIVE:

(I) OK;

$$\begin{array}{lcl} \mathrm{P}(E) > 1 & \Longrightarrow & \text{pay more than } M \text{ to win } M\ ! \\ \mathrm{P}(E) < 0 & \Longrightarrow & \text{paid to play !} \end{array}$$

(II) OK; if $E = \Omega$, must pay M so that neither side will be bound to lose.

(III) OK; Consider two bets on events E and F,

$$\left.\begin{array}{lcl} \mathrm{P}(E) = p_E & \Longrightarrow & \text{pay } p_E M \text{ to win } M \text{ on } E \\ \mathrm{P}(F) = p_F & \Longrightarrow & \text{pay } p_F M \text{ to win } M \text{ on } F \end{array}\right\} \Longrightarrow \text{pay } (p_E + p_F)M \text{ to win } M \text{ on } E \text{ or } F$$

if E and F are mutually exclusive, so $\mathrm{P}(E \cup F) = (p_E + p_F) = \mathrm{P}(E) + \mathrm{P}(F)$.

2. Free choice, but suggest

   (a) $\Omega = \{\mathrm{JAN}, \mathrm{FEB}, ..., \mathrm{DEC}\}$; equally likely outcomes ?

   (b) $\Omega = \{x : 1889 \le x \le 2000\}$; relative frequency, subjective ?

   (c) $\Omega = \{1001, 1002, ..., 5999, 6000\}$; equally likely outcomes calculations give probabilities, but these outcomes are not equally likely.

   (d) $\Omega = \{x : x \ge 0\}$; relative frequency ?

   (e) $\Omega = \{x : -50 \le x \le 35\}$; relative frequency, subjective ?