M1S : EXERCISE SHEET 10 : SOLUTIONS

1. Joint mass function: by inspection (i.e. by counting sample outcomes)

		X						$f_Y(y)$
		1	2	3	4	5	6	
	1	p	$2p$	$2p$	$2p$	$2p$	$2p$	$11p$
	2	0	p	$2p$	$2p$	$2p$	$2p$	$9p$
Y	3	0	0	p	$2p$	$2p$	$2p$	$7p$
	4	0	0	0	p	$2p$	$2p$	$5p$
	5	0	0	0	0	p	$2p$	$3p$
	6	0	0	0	0	0	p	p
	$f_X(x)$	p	$3p$	$5p$	$7p$	$9p$	$11p$	

where $p = 1/36$. Column totals specify $f_X(x)$ obtained from the joint mass function by summation over range of values of Y. Similarly, row totals give $f_Y(y)$.

Conditional mass functions given by rows/columns divided by their totals. For example, consider the conditional mass function for X, given that $Y = y$ defined by

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\mathrm{P}[X = x, Y = y]}{\mathrm{P}[Y = y]}.$$

To calculate the conditional mass function for X, given that $Y = 1$, say, take the row corresponding to $Y = 1$, and divide by that row total ($11p$), giving

$$f_{X|Y}(1|1) = 1/11 \quad f_{X|Y}(2|1) = 2/11 \quad f_{X|Y}(2|1) = 2/11$$

$$f_{X|Y}(4|1) = 2/11 \quad f_{X|Y}(5|1) = 2/11 \quad f_{X|Y}(6|1) = 2/11$$

X and Y are not independent as, for at least one pair (x, y), $f_{X,Y}(x, y) \neq f_X(x) f_Y(y)$.

2. For pairs (x, y), $f_{X,Y}(x, y) = \mathrm{P}[X = x, Y = y] = n_{xy}/n$ say, where $n = \binom{52}{2}$ (number of ways of choosing two cards from 52). Using combinatorics arguments,

$$n_{xy} = \begin{cases} \binom{44}{2} & x = y = 0 \\ \binom{44}{1}\binom{4}{1} & x = 1, y = 0 \text{ or } x = 0, y = 1 \\ \binom{4}{2} & x = 2, y = 0 \text{ or } x = 0, y = 2 \\ \binom{4}{1}\binom{4}{1} & x = y = 1 \end{cases}$$

Hence the joint mass function is specfied by

		X		
		0	1	2
	0	0.7134	0.1327	0.0045
Y	1	0.1327	0.0121	0
	2	0.0045	0	0

Marginal mass functions f_X and f_Y calculated as row/column totals, and X and Y are not independent as, for at least one pair (x, y), $f_{X,Y}(x, y) \neq f_X(x) f_Y(y)$.

3. $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^{\infty} y^{-1} e^{-y-x/y}\ dx = e^{-y},\ y > 0, \Longrightarrow Y \sim Exponential(1).$

4. Note that joint density is only non-zero when $0 \leq y \leq x \leq 1$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{x} x^{-1}\ dy = 1 \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_y^{1} x^{-1}\ dx = -\log y \quad 0 \leq y \leq 1$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = x^{-1} \quad 0 \leq y \leq x.$$

$$\mathrm{P}[\ Y \leq y \mid X = x\ ] = F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(t|x)\ dt = \int_0^{y} x^{-1} dt = y/x \quad 0 \leq y \leq x.$$

5.(i) Need $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = 1$. Hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = \int_0^1\int_0^1 c(x+2y)\ dxdy = 1 \Longrightarrow c = 2/3.$$

(ii) By definition

$$\begin{aligned}\mathrm{P}[\ X \leq 1/2,\ Y \leq 1/3\ ] &= F_{X,Y}(1/2,1/3) = \int_{-\infty}^{1/3}\int_{-\infty}^{1/2} f_{X,Y}(x,y)\ dxdy \\ &= \int_0^{1/3}\int_0^{1/2} 2(x+2y)/3\ dxdy = 7/108.\end{aligned}$$

(iii) Marginal densities

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dy = \int_0^{1} 2(x+2y)/3\ dy = 2(x+1)/3 \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dx = \int_0^{1} 2(x+2y)/3\ dx = (4y+1)/3 \quad 0 \leq y \leq 1$$

(iv) By definition

$$\begin{aligned}\mathrm{P}[\ Y \leq 1/3 \mid X \leq 1/2\ ] &= \frac{\mathrm{P}[\ X \leq 1/2,\ Y \leq 1/3]}{\mathrm{P}[\ X \leq 1/2\ ]} = \frac{F_{X,Y}(1/2,1/3)}{F_X(1/2)} \\ &= \frac{7/108}{\int_0^{1/2} f_X(x)\ dx} = \frac{7/108}{\int_0^{1/2} 2(x+1)/3\ dx} = \frac{7/108}{45/108} = \frac{7}{45}\end{aligned}$$

6.(i) $F_{X_i}(x) = 1 - e^{-\lambda_i x}$ for $x \geq 0$, $i = 1, 2$. Then if $U = \text{Min}\{X_1, X_2\}$, for $u \geq 0$,

$$\begin{aligned} F_U(u) &= \text{P}[\, U \leq u \,] = \text{P}[\, \text{Min}\{X_1, X_2\} \leq u \,] \\ &= 1 - \text{P}[\, \text{Min}\{X_1, X_2\} > u \,] = 1 - \text{P}[\, X_1 > u, X_2 > u \,] = 1 - \text{P}[\, X_1 > u \,]\text{P}[\, X_2 > u \,] \\ &= 1 - (1 - \text{P}[\, X_1 \leq u \,])(1 - \text{P}[\, X_2 \leq u \,]) = 1 - e^{-\lambda_1 u} e^{-\lambda_2 u} \\ &= 1 - e^{-(\lambda_1 + \lambda_2) u} \end{aligned}$$

Hence $U \sim \mathit{Exponential}(\lambda_1 + \lambda_2)$.

(ii) If $V = \text{Max}\{X_1, X_2\}$, then for $v \geq 0$,

$$\begin{aligned} F_V(v) &= \text{P}[\, V \leq v \,] = \text{P}[\, \text{Max}\{X_1, X_2\} \leq v \,] \\ &= \text{P}[\, X_1 \leq v, X_2 \leq v \,] = \text{P}[\, X_1 \leq v \,]\text{P}[\, X_2 \leq v \,] \\ &= (1 - e^{-\lambda_1 v})(1 - e^{-\lambda_2 v}) \end{aligned}$$

(iii) If $T = X_1 + X_2$, then the range $\mathbb{T} = \mathbb{R}$, and so for $t > 0$, let A_t be the region of $\mathbb{R}^2$ defined by

$$A_t = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq t\}$$

that is, the triangular region bounded by the three lines $x_1 = 0$, $x_2 = 0$ and $x_1 + x_2 = t$. Then $\text{P}[\, T \leq t \,] = \text{P}[\, (X_1, X_2) \in A_t \,]$, and hence

$$\begin{aligned} F_T(t) &= \int_{A_t}\!\!\int f_{X_1,X_2}(x_1, x_2)\; dx_1 dx_2 \;=\; \int_0^t \left\{ \int_0^{t-x_2} f_{X_1}(x_1) f_{X_2}(x_2)\; dx_1 \right\} dx_2 \\ &= \int_0^t \left\{ \int_0^{t-x_2} \lambda_1 e^{-\lambda_1 x_1}\; dx_1 \right\} \lambda_2 e^{-\lambda_2 x_2} dx_2 \;=\; \int_0^t (1 - e^{-\lambda_1 (t - x_2)}) \lambda_2 e^{-\lambda_2 x_2} dx_2 \\ &= \int_0^t \lambda_2 e^{-\lambda_2 x_2} dx_2 - \int_0^t \lambda_2 e^{-(\lambda_1 (t - x_2) + \lambda_2 x_2)} dx_2 \\ &= (1 - e^{-\lambda_2 t}) - \lambda_2 e^{-\lambda_1 t} \int_0^t e^{-(\lambda_2 - \lambda_1) x_2} dx_2 \end{aligned}$$

Hence, for $t > 0$,

$$F_T(t) = \begin{cases} 1 - \dfrac{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1} & \lambda_1 \neq \lambda_2 \\[2ex] 1 - (1 + \lambda t) e^{-\lambda t} & \lambda_1 = \lambda_2 = \lambda \end{cases}$$

and so by differentiation, for $t > 0$,

$$f_T(t) = \begin{cases} \dfrac{\lambda_1 \lambda_2 (e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2 - \lambda_1} & \lambda_1 \neq \lambda_2 \\[2ex] \lambda^2 t e^{-\lambda t} & \lambda_1 = \lambda_2 = \lambda \end{cases}$$

Note that this result is a special case of the convolution theorem, and the solution indicates how the derivation of the convolution theorem works.

7. Assuming that X and Y are independent in each case;

BINOMIAL: For $t = 0, 1, ..., n_X + n_Y$,

$$\begin{aligned} f_T(t) &= \sum_{x=-\infty}^{\infty} f_X(x) f_Y(t-x) = \sum_{x=0}^{t} \binom{n_X}{x} \theta^x (1-\theta)^{n_X - x} \binom{n_Y}{t-x} \theta^{t-x} (1-\theta)^{n_Y - t + x} \\ &= \sum_{x=0}^{t} \binom{n_X}{x} \binom{n_Y}{t-x} \theta^t (1-\theta)^{n_X + n_Y - t} = \binom{n_X + n_Y}{t} \theta^t (1-\theta)^{n_X + n_Y - t} \end{aligned}$$

Hence $T \sim Bin(n_X + n_Y, \theta)$.

POISSON: For $t = 0, 1, 2, ...,$

$$\begin{aligned} f_T(t) &= \sum_{x=-\infty}^{\infty} f_X(x) f_Y(t-x) = \sum_{x=0}^{t} \frac{\lambda_1{}^x}{x!} e^{-\lambda_1} \frac{\lambda_2{}^{t-x}}{(t-x)!} e^{-\lambda_2} \\ &= \frac{(\lambda_1 + \lambda_2)^t}{t!} e^{-(\lambda_1 + \lambda_2)} \left[\sum_{x=0}^{t} \binom{t}{x} \left\{ \frac{\lambda_1}{\lambda_1 + \lambda_2} \right\}^x \left\{ \frac{\lambda_2}{\lambda_1 + \lambda_2} \right\}^{t-x} \right] = \frac{(\lambda_1 + \lambda_2)^t}{t!} e^{-(\lambda_1 + \lambda_2)} \end{aligned}$$

Hence $T \sim Poisson(\lambda_1 + \lambda_2)$.

GEOMETRIC: For $t = 2, 3, ...,$

$$f_T(t) = \sum_{x=-\infty}^{\infty} f_X(x) f_Y(t-x) = \sum_{x=1}^{t-1} (1-\theta)^{x-1} \theta \; (1-\theta)^{t-x-1} \theta = (t-1) \theta^2 (1-\theta)^{t-2}$$

Hence $T \sim NegBinomial(2, \theta)$.

EXPONENTIAL: For $t > 0$,

$$f_T(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t-x) \; dx = \int_0^t \lambda e^{-\lambda x} \lambda e^{-\lambda(t-x)} \; dx = \lambda^2 t e^{-\lambda t}$$

Hence $T \sim Gamma(2, \lambda)$.

NORMAL:

$$\begin{aligned} f_T(t) &= \int_{-\infty}^{\infty} f_X(x) f_Y(t-x) \; dx \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma_X^2} \right)^{1/2} \exp\left\{ -\frac{1}{2\sigma_X^2} (x - \mu_X)^2 \right\} \left(\frac{1}{2\pi\sigma_Y^2} \right)^{1/2} \exp\left\{ -\frac{1}{2\sigma_Y^2} (t - x - \mu_Y)^2 \right\} \; dx \\ &= \left(\frac{1}{2\pi\sigma^2} \right)^{1/2} \exp\left\{ -\frac{1}{2\sigma^2} (t - \mu)^2 \right\} \end{aligned}$$

where $\mu = \mu_X + \mu_Y$ and $\sigma^2 = \sigma_X^2 + \sigma_Y^2$; the result follows by completing the square in x in the exponential term, using

$$\frac{1}{\sigma_X^2} (x - \mu_X)^2 + \frac{1}{\sigma_Y^2} (t - x - \mu_Y)^2 = \frac{\sigma^2}{\sigma_X^2 \sigma_Y^2} \left(x - \frac{\sigma_2 \mu - \sigma_X^2 t}{\sigma^2} \right)^2 + \frac{1}{\sigma^2} (t - \mu)^2$$

Hence $T \sim Normal(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

8. (i) Let X_i be the ith observed score, for $i = 1, ..., 20$. Then $X_1, ..., X_{20}$ are independent and identical random variables, and

$$
\begin{aligned}
\mathrm{E}_{f_{X_i}}[X_i] &= \sum_{i=1}^{6} i/6 = 7/2 = \mu \\
\mathrm{E}_{f_{X_i}}[X_i^2] &= \sum_{i=1}^{6} i^2/6 = 91/6 \Longrightarrow \mathrm{Var}_{f_{X_i}}[X_i] = \mathrm{E}_{f_{X_i}}[X_i^2] - \left\{\mathrm{E}_{f_{X_i}}[X_i]\right\}^2 = 35/12 = \sigma^2
\end{aligned}
$$

If $T = X_1 + ... + X_{20}$, then the central limit theorem implies that

$$F_T(t) = \mathrm{P}[\, T \leq t \,] \approx \Phi\left(\frac{t - n\mu}{\sqrt{n\sigma^2}}\right)$$

or, approximately, $T \sim N(n\mu, n\sigma^2)$. Hence

$$\mathrm{P}[\, 30 \leq T \leq 40 \,] \approx \Phi\left(\frac{40 - 70}{\sqrt{350/6}}\right) - \Phi\left(\frac{30 - 70}{\sqrt{350/6}}\right)$$

(ii) In each case, looking for a representation of random variable X say as the sum of independent and identical random variables $X_1, ..., X_n$. By the results in question 7, this can be achieved for many standard distributions; for example

X	X_i
$Binomial(n, \theta)$	$Bernoulli(\theta)$
$Poisson(\lambda)$	$Poisson(\lambda/n)$
$NegBinomial(n, \theta)$	$Geometric(\theta)$
$Gamma(n, \lambda)$	$Exponential(\lambda)$

In each case, can calculate $\mu = \mathrm{E}_{f_{X_i}}[X_i]$ and $\sigma^2 = \mathrm{Var}_{f_{X_i}}[X_i]$ easily, and hence use central limit theorem formula to approximate the distribution function of X; not that we do not have a closed form representation for the distribution function for any of these standard distributions, and hence the central limit theorem approximation is potentially useful.