M1S : ASSESSED COURSEWORK 3 : SOLUTIONS

1. (i) To deduce $\mathbb{X}$, note first that we require $f_X(x)$ to be non-negative. The function $-\ln x$ is only defined when $x > 0$, and is only non-negative when $x \leq 1$. Hence $\mathbb{X} \equiv (0,1]$ is largest potential range. Now we also require that

$$\int_{\mathbb{X}} f_X(x)\, dx = 1$$

and here

$$\begin{aligned}\int_{\mathbb{X}} f_X(x)\, dx &= \int_0^1 f_X(x)\, dx = \int_0^1 -\ln x\, dx = \int_0^1 -1.\ln x\, dx \\ &= [-x\ln x]_0^1 + \int_0^1 x\frac{1}{x}\, dx \qquad \text{(by parts)} \\ &= 0 + 1 = 1\end{aligned}$$

so the function integrates to 1 on $\mathbb{X} \equiv (0,1]$ Hence we deduce that $(0,1]$ is the appropriate range for the random variable, and we have by the identical argument that

$$F_X(x) = \int_0^x f_X(t)\, dt = \int_0^x -\ln t\, dt = x(1-\ln x) \qquad 0 < x \leq 1$$

[5 MARKS]

(ii) We have that

$$h_X(x) = \frac{f_X(x)}{1-F_X(x)} \qquad x > 0$$

which we may re-express, setting $y = F_X(x)$

$$h_X(x) = \frac{1}{1-y}\frac{dy}{dx} \tag{1}$$

as, from the definition given we know that

$$\frac{dy}{dx} = \frac{d}{dx}F_X(x) = f_X(x)$$

Integrating both sides of (1) we have

$$H_X(x) = \int \frac{1}{1-y}\frac{dy}{dx}dx = -\log(1-y)$$

where $H_X(x)$ is the integral of function $h_X(x)$. On rearrangement,

$$y = F_X(x) = 1 - \exp\{-H_X(x)\}$$

from which we deduce finally that

$$F_X(x) = 1 - \exp\left\{-\int_0^x h_X(t)\, dt\right\}$$

More formally,

$$h_X(x) = \frac{f_X(x)}{1-F_X(x)} \Longrightarrow H_X(x) = \int_0^x h_X(t)\, dt = \int_0^x \frac{f_X(t)}{1-F_X(t)}\, dt = [-\log(1-F_X(t)]_0^x = -\log(1-F_X(x))$$

and the results follows on rearrangement.

[5 MARKS]

2. We have for the cdf of X, for time $x > 0$

$$
\begin{aligned}
F_X(x) &= \mathrm{P}[X \le x] = 1 - \mathrm{P}[X > x] \\
&= 1 - \mathrm{P}[n\text{th event occurs } \textit{later} \text{ than time } x] \\
&= 1 - \mathrm{P}[\textit{fewer} \text{ than } n \text{ events occur in } [0, x)]]] \\
&= 1 - \sum_{i=0}^{n-1} \frac{e^{-\lambda x}(\lambda x)^i}{i!}
\end{aligned}
$$

using the Poisson process model assumptions. On diffferentiation to get $f_X(x)$, we may differentiate term by term in the summation

$$
\begin{aligned}
f_X(x) &= -\sum_{i=0}^{n-1} \frac{\lambda^i}{i!} \frac{d}{dx} \left\{ x^i e^{-\lambda x} \right\} \\
&= -\sum_{i=0}^{n-1} \frac{\lambda^i}{i!} \left\{ i x^{i-1} e^{-\lambda x} - \lambda x^i e^{-\lambda x} \right\} \\
&= -e^{-\lambda x} \left[\lambda \sum_{i=1}^{n-1} \frac{(\lambda x)^{i-1}}{(i-1)!} - \lambda \sum_{i=0}^{n-1} \frac{(\lambda x)^i}{i!} \right]
\end{aligned}
$$

splitting the sum in two parts, noting that the first term in the first sum ($i = 0$) does not contribute to the sum, and then combining terms. Now note that

$$
\sum_{i=1}^{n-1} \frac{(\lambda x)^{i-1}}{(i-1)!} = \sum_{j=0}^{n-2} \frac{(\lambda x)^j}{j!}
$$

where $j = i - 1$, and so

$$
\begin{aligned}
f_X(x) &= -\lambda e^{-\lambda x} \left[\sum_{j=0}^{n-2} \frac{(\lambda x)^j}{j!} - \sum_{i=0}^{n-1} \frac{(\lambda x)^i}{i!} \right] \\
&= \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}
\end{aligned}
$$

as other terms cancel. Hence

$$
f_X(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} \qquad x > 0
$$

as required.

[10 MARKS]