UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2000

This paper is also taken for the relevant examination for the Associateship.

M1S PROBABILITY AND STATISTICS I

DATE: Monday, 22th May 2000 TIME: 10 am – 12 noon

Credit will be given for all questions attempted, but extra credit will be given for complete, or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

1. *a)* State the Three Axioms of Probability.

If events E and F are general events in sample space Ω, prove that

i) $E \subseteq F \implies \mathrm{P}(E) \leq \mathrm{P}(F)$,

ii) $\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F)$.

Consider a countable collection of events $E_1, E_2, \ldots$ that form a partition of Ω. Prove that

$$\mathrm{P}(F) = \sum_{i=1}^{\infty} \mathrm{P}(F \cap E_i) .$$

b) In a tennis match between two players A and B, a single game is won by the first player to win four points, unless the score reaches three points all, in which case, the game is only won when a player achieves a lead of two points.

It is assumed that, when A is serving, the probability that A wins any point is p. It is also assumed that all points are mutually independent.

Let W_i be the event that A wins a service game in precisely i points, and let D be the event that the game reaches three points all.

i) Show that

$$p_D = \mathrm{P}(D) = \binom{6}{3} p^3 (1-p)^3 .$$

ii) Find the probability that A wins a service game in six or fewer points.

iii) Show that the probability that A wins a service game in precisely $6+2i$ points (for $i = 1, 2, \ldots$), $\mathrm{P}(W_{6+2i})$, is given by

$$\mathrm{P}(W_{6+2i}) = \{2p(1-p)\}^{i-1} p^2 p_D .$$

Hence find the probability that A wins a service game.

2. *a)* For two events E and F, with $\mathrm{P}(F) > 0$, define the *conditional probability* of E given F.

i) Prove that if E_1 and E_2 are exclusive events

$$\mathrm{P}(E_1 \cup E_2 \,|\, F) \;=\; \mathrm{P}(E_1 \,|\, F) \;+\; \mathrm{P}(E_2 \,|\, F) \;.$$

ii) State and prove the general version of Bayes Theorem in relation to a finite partition $E_1, \ldots, E_k$ of Ω and event $F \subset \Omega$, where $\mathrm{P}(E_i) > 0$ for $i = 1, 2, \ldots, k$.

b) Image analysis is to be used to classify each digit in a handwritten telephone number.

In any analysis, the probability that a digit $i \in \{0, \ldots, 9\}$ is correctly identified is α_i. However, the digit may be misclassified as $j \neq i$ with probability β_{ji}, where

$$\alpha_i \;+\; \sum_{j \neq i} \beta_{ji} \;=\; 1$$

for $i = 0, 1, \ldots, 9$.

Before the results of the image analysis of any individual digit are known, it is assumed that the digit under investigation is i with probability p_i, for $i = 0, 1, \ldots, 9$.

For $i = 0, 1, \ldots, 9$, let C_i be the event that a digit under study is classified i, and let D_i be the event that a digit is actually i.

Using this notation, find

i) the probability that the first digit is classified as a 9,

ii) the conditional probability that the first digit is actually a 9, given that the image analysis classifies it as a 9.

Given the result of the image analysis of a single digit, which is the *most probable* actual value of the digit under study?

3. *a)* *i)* Prove, by using a binary sequence representation, or otherwise, that the number of distinct ways of allocating r indistinguishable objects to n boxes is

$$\binom{n+r-1}{r} .$$

ii) A class of $4n$ people comprises $3n$ of female students and n male students. A representative panel of $4k$ ($k < n$) students is to be selected.

If the panel is required to comprise females and males in a ratio of 3:1, find the *number* of different panels that may be selected.

If the panel is selected from the class so that all subsets of $4k$ from $4n$ individuals are equally likely, find the *probability* that the selected panel comprises females and males in the ratio 3:1.

b) A referendum is to be carried out in a population of N eligible voters who will either vote "YES" (in favour of the motion) or "NO" (against the motion). An opinion poll of a randomly selected group of n voters indicated r "YES" voters and $n - r$ "NO" voters.

Find an expression for the conditional probability that number, R, of voters in the population who will vote "YES" is equal to x, for an appropriate range of values of x, *given* the result of the opinion poll.

[*Assume that voting intentions are fixed for all members of the population, that each individual polled answers truthfully, and that all samples of size n from N are equally likely to be selected for polling. Recall that the Hypergeometric formula for suitably defined N, R, n, r is*

$$\frac{\binom{n}{r}\binom{N-n}{R-r}}{\binom{N}{R}} = \frac{\binom{R}{r}\binom{N-R}{n-r}}{\binom{N}{n}}$$

when each combination term is defined.]

4. *a)* Continuous random variable X with range $\mathbb{X} = \mathbb{R}$ has probability density function (pdf) f_X defined by

$$f_X(x) = \frac{c\, e^{\lambda x}}{(1 + e^{\lambda x})^2}$$

for parameter $\lambda > 0$.

i) Find the value of constant c.

ii) Find the cumulative distribution (cdf), F_X, corresponding to this pdf.

iii) Find the pdf, f_Y, of random variable Y defined by $Y = e^{\lambda X}$.

b) Consider now the random variable U defined by

$$U \;=\; \frac{Y}{1+Y} \quad .$$

Find the cdf of U.

For continuous random variable V with cdf F_V, find the cdf of the random variable $W = F_V(V)$.

[*Assume that, for* $0 \le w \le 1$, *there is a unique solution to* $F_V(v) = w$.]

5. *a)* For discrete random variable X with mass function f_X and finite range $\mathbb{X} = \{\nvdash, \nVdash, \nvDash, \ldots, \ltimes\}$, define the *probability generating function* (pgf) of X, G_X.

If $X \sim Bernoulli(\theta)$, so that

$$f_X(x) \;=\; \theta^x(1-\theta)^{1-x} \qquad x = 0, 1$$

and zero otherwise, find the pgf of X.

Suppose now that $X_1, \ldots, X_n$ are independent and identically distributed $Bernoulli(\theta)$ random variables. Let random variable Y be defined by

$$Y = \sum_{i=1}^{n} X_i \quad .$$

Show that the pgf of Y, G_Y, is $\;G_Y(t) \;=\; (1-\theta+\theta t)^n$.
[*You may quote without proof any general pgf results that you use.*]

By writing G_Y as a power series in t, find the probability mass function of Y, f_Y.

b) For a general pgf G_X, let $G_X^{(r)}(t)$ be defined by

$$G_X^{(r)}(t) \;=\; \frac{d^r}{ds^r}\left\{G_X(s)\right\}_{s=t}$$

for $r = 1, 2, \ldots$, that is, the rth derivative of G_X evaluated at t.

Show that

$$G_X^{(r)}(1) \;=\; \mathrm{E}_{f_X}\left[\,X(X-1)(X-2)\cdots(X-r+1)\,\right] \quad .$$

Hence, or otherwise, find the *expectation* of Y (defined in *a)*), and show that the *variance* of Y is $n\theta(1-\theta)$.