GENERATING FUNCTIONS : SOLUTION OF COMBINATORIAL AND OTHER PROBLEMS

Introduction

The general theme behind the use of Generating Functions as a means of constructing solutions to problems in combinatorics is to attempt to identify a function g of dummy variable x that has a power series representation in which the coefficients of the increasing powers of x can be identified as the required solutions. Below, we will see how to construct solutions to specific combinatorics problems; first, we need some notation and general definitions.

DEFINITION

For a countable collection of real constants $\{a_r\} = \{a_0, a_1, a_2, ...\}$, the function $g(x)$ as a power series in x, defined by

$$g(x) = a_0 + a_1 x + a_2 x^2 + ... + a_r x^r + ...$$

is a *generating function* for $\{a_r\}$.

A simple example of a generating function is the function $g(x) = (1+x)^n$, which we may express using a binomial expansion as

$$g(x) = (1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + ... + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

in which the coefficient of x^r is $a_r = \binom{n}{r}$. Hence $g(x) = (1+x)^n$ is a generating function for the binomial coefficients; recall that these coefficients are the solutions to the simple combinatorics problem of enumerating the number of ways of selecting an (unordered) sample of r objects from n without replacement.

We attempt to construct similar generating functions for more complex enumeration problems. To motivate our construction, consider the above example; we have

$$g(x) = (1+x)^n = (1+x)(1+x)...(1+x)$$

that is, a product of n identical *polynomial factors*, each of which we may re-write $1 + x \equiv x^0 + x^1$. To identify the coefficient of x^r for $r = 0, 1, ..., n$ in the expansion, we note that a general term x^r arises when we take one term from each factor and multiply the terms together. Let the exponent of the term taken from bracket i be denoted e_i, for $i = 1, ..., n$. Then a term x^r only arises if

$$x^{e_1} x^{e_2} ... x^{e_n} = x^r$$

or equivalently, by equating exponents, if

$$e_1 + e_2 + ... + e_n = r$$

where $0 \leq e_i \leq 1$ for each i. Hence the required coefficient is the number of distinct solutions to this equation.

Clearly, the dual representation extends easily to more general power series; consider the function

$$g(x) = (1+x+x^2)^n = (1+x+x^2)(1+x+x^2)...(1+x+x^2).$$

The coefficient of x_r is given by enumeration of the solutions to the equation

$$e_1 + e_2 + ... + e_n = r$$

where $0 \leq e_i \leq 2$ for each i.

For example, if $n = 4$ and $r = 7$, we can readily identify the list of solutions to this equation as

e_1	e_2	e_3	e_4
2	2	2	1
2	2	1	2
2	1	2	2
1	2	2	2

In general, however, listing the solutions in this form may a formidable task; fortunately, we will only rarely have to do so.

Note that, in general, if we have a generating function $g(x)$ defined by as a product of n polynomial factors comprising $r_1, r_2, ..., r_n$ terms respectively, for example

$$g(x) = \prod_{i=1}^{n} (1 + x + x^2 + ... + x^{r_i})$$

then the total number of terms in the expansion is

$$r_1 \times r_2 \times ... \times r_n$$

and the coefficient of x^r is the number of solutions to

$$e_1 + e_2 + ... + e_n = r \qquad \text{where } 0 \leq e_i \leq r_i$$

Constructing the Solutions to Combinatorics Problems

The general framework above allows us to construct generating functions that provide solution to combinatorics problems, specifically complex partition, finite population, and occupancy problems, in a routine fashion. We illustrate the method by means of some simple examples.

EXAMPLE 1

Consider a population of $N = 12$ objects, comprised of equal numbers of four Types. We wish to enumerate the *distinct* samples of r objects from the population. Consider a typical sample, and let e_i represent the number of objects of Type i (for $i = 1, 2, 3, 4$). Then the number, a_r, of distinct samples is the number of solutions to

$$e_1 + e_2 + e_3 + e_4 = r$$

where $0 \leq e_i \leq 3$. By comparison with the previous section, we may immediately write down the generating function for $\{a_r\}$ as

$$g(x) = (1 + x + x^2 + x^3)(1 + x + x^2 + x^3)(1 + x + x^2 + x^3)(1 + x + x^2 + x^3) = (1 + x + x^2 + x^3)^4$$

in which we wish to identify the coefficient of x^r. The four polynomial factors correspond to the four Types, and each contains the four terms x^0, x^1, x^2, x^3 for the possible numbers $0, 1, 2, 3$ of objects of that Type that have been selected.

Computing the required coefficient by multiplying out the polynomial factors is not necessarily a straightforward procedure, but we will see some techniques later via which the computation may be simplified.

EXAMPLE 2
To enumerate the numbers of ways of selecting $r = 6$ objects from a population comprising three Types of object, with replacement, such that no more than four objects of one Type are selected, we may re-express the problem as finding the solutions to

$$e_1 + e_2 + e_3 = 6$$

where $0 \leq e_i \leq 4$ for $i = 1, 2, 3$ are again the number of objects of each Type selected. Hence a generating function for this problem is

$$g(x) = (1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4)(1 + x + x^2 + x^3 + x^4) = (1 + x + x^2 + x^3 + x^4)^3$$

in which we wish to identify the coefficient of x^6. If the problem were re-stated to allow any number (rather than at most four) of objects from a particular Type, then the appropriate generating function would be the product

$$g(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + ...)^3$$

that is, with each polynomial factor being an infinite sum of terms.

EXAMPLE 3
Consider the occupancy/distribution problem in which r indistinguishable balls are to allocated to three boxes with at most five balls in the first box. Writing e_i for the number of of balls allocated to box $i = 1, 2, 3$, we seek the solutions of

$$e_1 + e_2 + e_3 = r$$

where $0 \leq e_1 \leq 5$, but $0 \leq e_2, e_3 \leq r$ Hence the corresponding generating function is

$$g(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + ... + x^r)^2$$

with the three polynomial factors corresponding to the three boxes, and the terms in each factor corresponding to the numbers of balls that may be allocated to that box. Note that, as there are no conditions attached to boxes two and three, we can essentially choose e_2 and e_3 to be of any size, so that

$$g(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + ...)^2$$

is also an acceptable generating function; we have an implicit constraint in the integer equation that all of e_1, e_2 and e_3 are in the range from 0 to r.

EXAMPLE 4
Consider the occupancy/distribution problem in which r indistinguishable balls are to allocated to five boxes with an even number less than or equal to ten balls in each of the first two boxes, and between three and five balls in the the remaining boxes. Writing e_i for the number of balls allocated to box $i = 1, 2, 3, 4, 5$, we now seek the solutions of

$$e_1 + e_2 + e_3 + e_4 + e_5 = r$$

where $0 \leq e_1, e_2 \leq 10$, and $3 \leq e_3, e_4, e_5 \leq 5$. Hence the corresponding generating function is

$$g(x) = (1 + x^2 + x^4 + x^6 + x^8 + x^{10})^2(x^3 + x^4 + x^5)^3$$

with the five polynomial factors corresponding to the boxes, and the terms in each factor corresponding to the numbers of balls that may be allocated to that box. Again, identifying the coefficient of x^r is not straightforward, but we have at least reduced the problem to one of algebraic manipulation.

Techniques for simplifying generating functions

To identify the required coefficient of a power of x in a generating function experssion, we will need to manipulate and simplify products of polynomial factors. In addition to the binomial expansion, the following simple results may be useful

Function	Expansion
$\dfrac{1-x^{m+1}}{1-x}$	$= 1+x+x^2+...+x^m$
$\dfrac{1}{1-x}$	$= 1+x+x^2+x^3+...+x^r+...$
$(1-x^m)^n$	$= 1-\binom{n}{1}x^m+\binom{n}{2}x^{2m}-...(-1)^n\binom{n}{n}x^{nm}$
$\dfrac{1}{(1-x)^n}$	$= 1+\binom{1+n-1}{1}x+\binom{2+n-1}{2}x^2+...\binom{r+n-1}{r}x^r+...$

Also, if we have two polynomial factors

$$f(x) = a_0 + a_1x + a_2x^2 + ... \qquad g(x) = b_0 + b_1x + b_2x^2 + ...$$

then the product

$$h(x) = f(x)g(x) = (a_0 + a_1x + a_2x^2 + ...)(b_0 + b_1x + b_2x^2 + ...)$$

has coefficient for x^r that is

$$a_0b_r + a_1b_{r-1} + ... + a_{r-1}b_1 + a_rb_0 = \sum_{j=0}^{r} a_jb_{r-j}$$

EXAMPLE

Consider allocating 25 indistinguishable balls to seven boxes, with no more than 10 balls in the first box, but with the numbers in the other boxes unconstrained. We seek solutions to the equation

$$e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 = 25$$

where $0 \le e_1 \le 10$, but $0 \le e_2, e_3, ..., e_7 \le r$ The corresponding generating function, using an infinite sum for each of the unconstrained boxes, is

$$g(x) = (1 + x + x^2 + x^3 + ... + x^{10})(1 + x + x^2 + ...)^6$$

which, using the identities above simplifies to

$$g(x) = \left(\frac{1-x^{11}}{1-x}\right)\left(\frac{1}{1-x}\right)^6 = (1-x^{11})\frac{1}{(1-x)^7}$$

We have, from the *negative binomial expansion* above that

$$(1-x)^{-7} = 1 + \binom{1+7-1}{1}x + \binom{2+7-1}{2}x^2 + ... + \binom{r+7-1}{r}x^r + ...$$

and thus terms in powers of x can be identified easily. In particular, the coefficient of x^{25} can be identified by noting

$$x^{25} = x^0 \times x^{25} + x^{11}x^{14}$$

so that the required coefficient of x^{25} is

$$a_{25} = 1 \times \binom{25+7-1}{25} + (-1) \times \binom{14+7-1}{14}$$

by combining the $x^0 = 1$ term from the first bracket with the x^{25} term from the second, and adding the x^{11} term from the first bracket multiplied the x^{14} term from the second.

Note that we can justify these terms using a direct combinatorics argument; we consider allocating the $r = 25$ balls to the $n = 7$ boxes without constraint which may be carried out in

$$\binom{n+r-1}{r} = \binom{7+25-1}{25}$$

ways, and then subtract the allocations that violate the constraint, that is, those allocations which place eleven or more balls in the first box; there are

$$\binom{(25-11)+7-1}{(25-11)}$$

such allocations, as we can first consider allocating the eleven balls to the first box, and then distributing the remaining fourteen throughout the boxes.

Partition Calculations

It is possible to use generating functions to construct the solution to partition problems. Typically, we wish to enumerate the ways of partitioning a number of r objects into subsets of different sizes. We equate this problem to one of expressing the integer r in terms of sums of smaller (positive) integers; for example, the integer 5 can be written as

$$\begin{array}{l} 1+1+1+1+1 \\ 1+1+1+2 \\ 1+2+2 \\ 1+1+3 \\ 2+3 \\ 1+4 \\ 5 \end{array}$$

and we enumerate the numbers of such expressions by labelling each sum according to the numbers of 1s, 2s, 3s, etc. that appear in the expression. We seek to find all (vector) solutions $(e_1, e_2, ..., e_r)$ to the equation

$$1e_1 + 2e_2 + 3e_3 + ... + re_r = r$$

so that the solutions listed above can be labelled $(5,0,0,0,0)$, $(3,1,0,0,0)$, $(1,2,0,0,0)$ etc. It can be shown that a suitable generating function for this problem is the product of polynomial factors

$$g(x) = \prod_{i=1}^{\infty} (1 + x^i + x^{2i} + x^{3i} + ...) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

which may be truncated at the rth term in the product for partitions of up to and including r objects.

For example, if $r = 3$, we have

$$g(x) = (1 + x + x^2 + x^3 + ...)(1 + x^2 + x^4 + x^6...)(1 + x^3 + x^6 + x^9 + ...)$$

where the first factor records the numbers of 1s, the second the number of 2s and the third the number of 3s.

The solutions to more specific partition problems may also be constructed using generating functions.

EXAMPLE
The number, a_r of ways of expressing r as a sum of distinct integers has generating function

$$g(x) = (1 + x)(1 + x^2)(1 + x^3)...(1 + x^k)...$$

where again the kth term corresponds to the number (either 0 or 1 as the integers used overall must be distinct) of ks that are used in the partition.

EXAMPLE
The number of ways of partitioning r into 2s, 3s, and 5s can be found using the generating function

$$g(x) = (1 + x^2 + x^4 + x^6...)(1 + x^3 + x^6 + x^9 + ...)(1 + x^5 + x^{10} + x^{15} + ...)$$

where the first factor records the numbers of 2s, the second the number of 3s and the third the number of 5s used in the partition.