TRANSFORMATIONS OF RANDOM VARIABLES: WORKED EXAMPLES

EXAMPLE

The maximum temperature in degrees Fahrenheit, X, measured in a type of chemical reaction varies between experiments according to a pdf f_X given by

$$f_X(x) = x \exp\left\{-\frac{1}{2}x^2\right\} \qquad x > 0$$

and zero otherwise. The maximum temperature measured in degrees Celsius, is a continuous random variable Y defined in terms of X by

$$Y = \frac{5}{9}(X - 32)$$

Note first that the range of the transformed variable is $\mathbb{Y} = \left\{y : y > -\frac{5}{9}32\right\}$ and the pdf of Y is computed by first inspecting the cdf of Y. From first principles,

$$\begin{aligned} F_Y(y) &= \mathrm{P}\left[Y \leq y\right] = \mathrm{P}\left[\frac{5}{9}(X - 32) \leq y\right] \\ &= \mathrm{P}\left[X \leq \frac{9}{5}y + 32\right] \\ &= F_X\left(\frac{9}{5}y + 32\right) \end{aligned}$$

On differentiation using the chain rule, and recalling that the derivative of the cdf is the pdf, we have

$$f_Y(y) = \frac{9}{5}f_X\left(\frac{9}{5}y + 32\right) = \frac{9}{5}\left(\frac{9}{5}y + 32\right)\exp\left\{-\frac{1}{2}\left(\frac{9}{5}y + 32\right)^2\right\} \qquad y > -\frac{5}{9}32$$

EXAMPLE

If continuous random variable U has a Uniform distribution on the interval $(0, 1)$, so that

$$f_U(u) = 1 \qquad F_U(u) = u \qquad 0 < u < 1$$

then to find the probability distribution of random variable X defined by

$$X = \frac{1}{\lambda}\log\left(\frac{U}{1 - U}\right)$$

we proceed as follows: By inspection, the range of the transformed variable is $(-\infty, \infty)$, and from first principles,

$$\begin{aligned} F_X(x) &= \mathrm{P}\left[X \leq x\right] \\ &= \mathrm{P}\left[\frac{1}{\lambda}\log\left(\frac{U}{1 - U}\right) \leq x\right] \\ &= \mathrm{P}\left[U \leq \frac{e^{\lambda x}}{1 + e^{\lambda x}}\right] \\ &= F_U\left(\frac{e^{\lambda x}}{1 + e^{\lambda x}}\right) = \frac{e^{\lambda x}}{1 + e^{\lambda x}} \end{aligned}$$

and hence on differentiation, we have

$$f_U(u) = \frac{\left(1 + e^{\lambda x}\right)\lambda e^{\lambda x} - e^{\lambda x}\lambda e^{\lambda x}}{\left(1 + e^{\lambda x}\right)^2} = \frac{\lambda e^{\lambda x}}{\left(1 + e^{\lambda x}\right)^2} \qquad x \in \mathbb{R}$$

EXAMPLE

If continuous random variable U has a Uniform distribution on the interval $(0,1)$, consider the random variable X defined by

$$X = 1 + \left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor$$

for parameter θ $(0 < \theta < 1)$, where $\lfloor a \rfloor$ is the integer part of a for real value a. The range of the transformed variable is the set

$$\mathbb{X} \equiv \{1, 2, 3, ...\}$$

and from first principles, for $x \in \mathbb{X}$

$$\begin{aligned}
F_X(x) &= \mathrm{P}[X \leq x] = \mathrm{P}\left[1 + \left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor \leq x\right] \\
&= \mathrm{P}\left[\left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor \leq x - 1\right] \\
&= \mathrm{P}\left[\left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor < x\right] && \text{as } x \text{ is integer-valued} \\
&= \mathrm{P}[\log U > x \log(1-\theta)] && \text{as } 0 < \theta < 1 \implies \log(1-\theta) < 0 \\
&= \mathrm{P}[U > (1-\theta)^x] = 1 - F_U((1-\theta)^x) \\
&= 1 - (1-\theta)^x && x = 1, 2, 3, ...
\end{aligned}$$

and hence

$$X \sim \mathit{Geometric}(\theta)$$

EXAMPLE

Random variable X measures the speed of a molecule of mass m in a gas at some temperature. Kinetic theory suggests that the pdf of X can be expressed as

$$f_X(x) = 4\sqrt{\frac{\lambda^3}{\pi}} x^2 \exp\left\{-\lambda x^2\right\} \qquad x > 0$$

for some constant $\lambda > 0$. The kinetic energy of the molecule is a continuous random variable Y defined by

$$Y = \frac{mX^2}{2}$$

The pdf of Y is computed as follows from the cdf; for $y > 0$

$$\begin{aligned}
F_Y(y) &= \mathrm{P}[Y \leq y] = \mathrm{P}\left[\frac{mX^2}{2} \leq y\right] \\
&= \mathrm{P}\left[X^2 \leq \frac{2y}{m}\right] \\
&= \mathrm{P}\left[-\sqrt{\frac{2y}{m}} \leq X \leq \sqrt{\frac{2y}{m}}\right] \\
&= \mathrm{P}\left[X \leq \sqrt{\frac{2y}{m}}\right] - \mathrm{P}\left[X < -\sqrt{\frac{2y}{m}}\right] = F_X\left(\sqrt{\frac{2y}{m}}\right)
\end{aligned}$$

as X is a **positive** random variable. Hence the pdf is obtained by differentiation as

$$f_Y(y) = \sqrt{\frac{2}{m}}\frac{1}{2}\frac{1}{\sqrt{y}}f_X\left(\sqrt{\frac{2y}{m}}\right) = \sqrt{\frac{1}{2my}}\,4\sqrt{\frac{\lambda^3}{\pi}}\left(\sqrt{\frac{2y}{m}}\right)^2\exp\left\{-\lambda\left(\sqrt{\frac{2y}{m}}\right)^2\right\}$$

$$= 4\sqrt{\frac{2\lambda^3}{\pi m^3}}y^{1/2}\exp\left\{-\frac{2\lambda}{m}y\right\}$$

That is, we have that, inspecting the terms in y

$$Y \sim Gamma\left(\frac{3}{2},\frac{2\lambda}{m}\right)$$

Note here that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ and

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{1/2-1}e^{-x}\,dx = \int_0^\infty \left(t^2\right)^{-1/2}e^{-t^2}\,(2t)\,dt = 2\int_0^\infty e^{-t^2}\,dt = \int_{-\infty}^\infty e^{-t^2}\,dt = \sqrt{\pi}$$

setting $t^2 = x$, and using the integral result from the Normal pdf proof.

EXAMPLE

A projectile is fired from the origin at velocity V and angle T from the horizontal. It lands a distance X away, where for gravitational constant g,

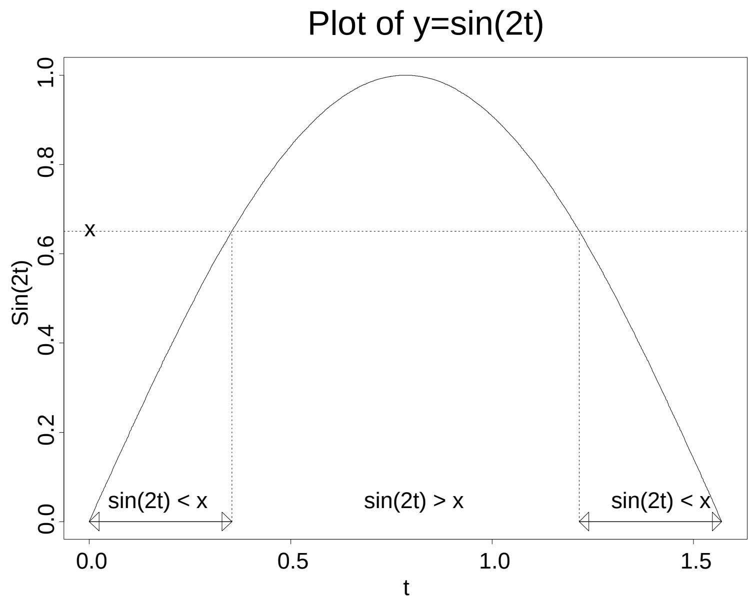
$$X = \frac{V^2}{g}\sin 2T$$

(i) If V is constant, but T has a Uniform distribution on $(0,\pi/2)$ then

$$f_T(t) = \frac{2}{\pi} \qquad 0 < t < \frac{\pi}{2} \qquad\qquad F_T(t) = \frac{2t}{\pi} \qquad 0 < t < \frac{\pi}{2}$$

The range of X is $\left(0,\frac{V^2}{g}\right)$, and for x in this range, the cdf of X is obtained as follows: by definition,

$$\mathrm{P}\left[X \le x\right] = \mathrm{P}\left[\frac{V^2}{g}\sin 2T \le x\right]$$



Hence, by inspection of the graph for sine, and consideration of the inverse sine function, we have that

$$\frac{V^2}{g}\sin 2T \leq x \qquad \Longleftrightarrow \qquad 2T \leq \sin^{-1}\left(\frac{gx}{V^2}\right) \quad \text{or} \quad 2T \geq \pi - \sin^{-1}\left(\frac{gx}{V^2}\right)$$

as the sin function is not 1-1. Hence

$$\mathrm{P}\left[X \leq x\right] = \mathrm{P}\left[2T \leq \sin^{-1}\left(\frac{gx}{V^2}\right)\right] + \mathrm{P}\left[2T \geq \pi - \sin^{-1}\left(\frac{gx}{V^2}\right)\right]$$

and so

$$F_X(x) = F_T\left(\frac{1}{2}\sin^{-1}\left(\frac{gx}{V^2}\right)\right) + 1 - F_T\left(\frac{1}{2}\left(\pi - \sin^{-1}\left(\frac{gx}{V^2}\right)\right)\right)$$

which simplifies to

$$F_X(x) = \frac{2}{\pi}\frac{1}{2}\sin^{-1}\left(\frac{gx}{V^2}\right) + 1 - \frac{2}{\pi}\frac{1}{2}\left(\pi - \sin^{-1}\left(\frac{gx}{V^2}\right)\right) = \frac{2}{\pi}\sin^{-1}\left(\frac{gx}{V^2}\right) \qquad 0 < x < \frac{V^2}{g}$$

On differentiation, we have the density of X as

$$f_X(x) = \frac{2}{\pi\sqrt{V^4/g^2 - x^2}} \qquad 0 < x < \frac{V^2}{g}$$

(ii) If T is constant, but V has density

$$f_V(v) = \frac{4}{\sqrt{\pi}}v^2\exp\left\{-v^2\right\} \qquad 0 < v$$

then X has range $(0, \infty)$, and

$$F_X(x) = \mathrm{P}\left[X \leq x\right] = \mathrm{P}\left[\frac{V^2}{g}\sin 2T \leq x\right] = \mathrm{P}\left[V^2 \leq \frac{gx}{\sin 2T}\right] = \mathrm{P}\left[V \leq \sqrt{\frac{gx}{\sin 2T}}\right] = F_V\left(\sqrt{\frac{gx}{\sin 2T}}\right)$$

and on differentiation, we have

$$f_X(x) = \frac{4}{\sqrt{\pi}}\frac{gx}{\sin 2T}\exp\left\{-\frac{gx}{\sin 2T}\right\} \cdot \sqrt{\frac{g}{\sin 2T}}\frac{1}{2}\frac{1}{\sqrt{x}} = \sqrt{\frac{4x}{\pi}}\sqrt[3]{\frac{g}{\sin 2T}}\exp\left\{-\frac{gx}{\sin 2T}\right\} \qquad 0 < x$$

Again

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp\left\{-\beta x\right\}$$

where

$$\alpha = \frac{3}{2} \qquad \beta = \frac{g}{\sin 2T}$$

as again $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$,and hence

$$X \sim Gamma\left(\frac{3}{2}, \frac{g}{\sin 2T}\right).$$

MATHEMATICAL BACKGROUND

Consider a discrete/continuous random variable X with range $\mathbb{X}$ and probability distribution described by mass/pdf f_X, or cdf F_X. Suppose g is a real-valued function whose domain includes $\mathbb{X}$, and suppose that

$$g: \begin{array}{l} \mathbb{X} \longrightarrow \mathbb{Y} \\ x \longmapsto y \end{array}$$

Then $Y = g(X)$ is also a random variable as Y is a function from Ω to $\mathbb{R}$.

Consider first the cdf of Y, F_Y, evaluated at a point $y \in \mathbb{R}$. We have

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \ \mathrm{P}[\ g(X) \leq y\ ] = \begin{cases} \displaystyle\sum_{x \in A_y} f_X(x) & \text{if } X \text{ is discrete} \\ \\ \displaystyle\int_{A_y} f_X(x)\ dx & \text{if } X \text{ is continuous} \end{cases}$$

where

$$A_y = \{\ x \in \mathbb{X}\ :\ g(x) \leq y\}$$

Attention thus centres on identifying, and computing the probability content of, the set is A_y.

1-1 TRANSFORMATIONS

The mapping $g(X)$ is a function of X from $\mathbb{X}$ which is **1-1 and onto** $\mathbb{Y}$ if,

(i) for each $x \in \mathbb{X}$, there exists one and only one y such that $y = g(x)$, and

(ii) for each $y \in \mathbb{Y}$, there exists an $x \in \mathbb{X}$ such that $g(x) = y$.

(in this context, g is onto $\mathbb{Y}$ by construction). If g is 1-1 then it is also a **monotonic** function on $\mathbb{X}$ and, crucially the inverse function g^{-1} is well-defined, that is, for unique values $x \in \mathbb{X}$ and $y \in \mathbb{Y}$

$$y = g(x) \qquad \Leftrightarrow \qquad g^{-1}(y) = x$$

The following theorem gives the distribution for random variable $Y = g(X)$ when g is 1-1.

THEOREM

Let X be a random variable with mass/density function f_X and support $\mathbb{X}$. Let g be a 1-1 function from $\mathbb{X}$ onto $\mathbb{Y}$ with inverse g^{-1}. Then $Y = g(X)$ is a random variable with support $\mathbb{Y}$ and

Discrete Case : The mass function of random variable Y is given by

$$f_Y(y) = f_X(g^{-1}(y)) \qquad y \in \mathbb{Y} = \{\ y \mid f_Y(y) > 0\ \}$$

where x is the unique solution of $y = g(x)$ (so that $x = g^{-1}(y)$).

Continuous Case : The pdf of random variable Y is given by

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dt} \left\{g^{-1}(t)\right\}_{t=y} \right| \qquad y \in \mathbb{Y} = \{\ y \mid f_Y(y) > 0\ \}$$

where $y = g(x)$, provided that the derivative $\dfrac{d}{dt}\left\{g^{-1}(t)\right\}$ is **continuous** and **non-zero** on $\mathbb{Y}$.

PROOF
Discrete case:

By direct calculation, $f_Y(y) = \mathrm{P}[\ Y = y\ ] = \mathrm{P}[\ g(X) = y\ ] = \mathrm{P}[\ X = g^{-1}(y)\ ] = f_X(x)$. where $x = g^{-1}(y)$, and hence $f_Y(y) > 0 \Longleftrightarrow f_X(x) > 0$.

Continuous case: Function g is either (I) a monotonic increasing, or (II) a monotonic decreasing function.

Case (I): If g is **increasing**, then for $x \in \mathbb{X}$ and $y \in \mathbb{Y}$, we have that $g(x) \leq y \Longleftrightarrow x \leq g^{-1}(y)$. Therefore, for $y \in \mathbb{Y}$,

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ g(X) \leq y\ ] = \mathrm{P}[\ X \leq g^{-1}(y)\ ] = F_X(g^{-1}(y))$$

and, by differentiation, because g is monotonic increasing,

$$f_Y(y) = f_X(g^{-1}(y))\frac{d}{dt}\left\{g^{-1}(t)\right\}_{t=y} = f_X(g^{-1}(y))\left|\frac{d}{dy}\left\{g^{-1}(y)\right\}_{t=y}\right| \qquad \text{as } \frac{d}{dt}\left\{g^{-1}(t)\right\} > 0.$$

Case (II): If g is **decreasing**, then for $x \in \mathbb{X}$ and $y \in \mathbb{Y}$ we have $g(x) \leq y \Longleftrightarrow x \geq g^{-1}(y)$. Therefore, for $y \in \mathbb{Y}$,

$$F_Y(y) = \mathrm{P}[\ Y \leq y\ ] = \mathrm{P}[\ g(X) \leq y\ ] = \mathrm{P}[\ X \geq g^{-1}(y)\ ] = 1 - F_X(g^{-1}(y))$$

so

$$f_Y(y) = -f_X(g^{-1}(y))\frac{d}{dt}\left\{g^{-1}(y)\right\} = f_X(g^{-1}(y))\left|\frac{d}{dt}\left\{g^{-1}(t)\right\}_{t=y}\right| \qquad \text{as } \frac{d}{dt}\left\{g^{-1}(t)\right\} < 0.$$

DEFINITION

Suppose transformation $g : \mathbb{X} \longrightarrow \mathbb{Y}$ is 1-1, and is defined by $g(x) = y$ for $x \in \mathbb{X}$. Then the **Jacobian** of the transformation, denoted $J(y)$, is given by

$$J(y) = \left|\frac{d}{dt}\left\{g^{-1}(t)\right\}_{t=y}\right|$$

that is, the absolute value of first derivative of g^{-1} evaluated at $y = g(x)$. Note that the inverse transformation $g^{-1} : \mathbb{Y} \longrightarrow \mathbb{X}$ has Jacobian $\dfrac{1}{J(x)}$

Note that the role of the Jacobian here is precisely the same as that of the "change of variables" term that appears in a substitution in an integral. That is, if X and Y are the two variables so that $Y = g(X)$, then by construction

$$\mathrm{P}\left[X \in A\right] \equiv \mathrm{P}\left[X \in B\right]$$

for sets $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$ where B is the image of A under g, $B \equiv \{y \in \mathbb{Y} : y = g(x) \text{ for some } x \in A\}$. Now, in the probability equation, introducing the pdfs for X and Y, we have

$$\int_A f_X(x)\, dx \equiv \int_B f_Y(y)\, dy$$

but if g is 1-1, we have, by changing variables in the left hand integral to $y = g(x)$ so that $x = g^{-1}(y)$ gives

$$\int_A f_X\left(g^{-1}(y)\right)\left|\frac{dx}{dy}\right|\, dy \equiv \int_A f_Y(y)\, dy$$

where $\left|\dfrac{dx}{dy}\right|$ is precisely the Jacobian term that appears above. Finally we can equate integrands, as this result holds for an **arbitrary** set A.