THE POISSON PROCESS

Consider an experiment involving events that occur repeatedly in time. Let $X(t)$ be the random variable representing the number of events that occur in the interval $[0,t)$, so that $X(t)$ takes values $0,1,2,....$ Suppose that

1. $X(0) = 0$
2. For all $0 < s \leq t$, $h > 0$, and non-negative integers n and m,

$$P[X(t+h) - X(t) = n | X(s) = m] = P[X(t+h) - X(t) = n]$$

   that is, the numbers of events occurring in disjoint intervals are probabilistically **independent**.
3. For $\triangle t > 0$ small,

$$P[X(t+\triangle t) - X(t) = 1] = \lambda \triangle t + O(\triangle t)$$

for some $\lambda > 0$, where

$$\lim_{\triangle t \longrightarrow 0} \frac{O(\triangle t)}{\triangle t} = 0$$

that is, the probability of **exactly one** event occurring in the small interval $[t, t+\triangle t)$ is, for small $\triangle t$, proportional to the length of the interval, $\triangle t$.

4. For $\triangle t > 0$ small,

$$P[X(t+\triangle t) - X(t) \geq 2] = O(\triangle t)$$

for some $\lambda > 0$, that is, the probability of **more than one** event occurring in a small interval $(t, t+\triangle t]$ is essentially zero.

Then, if

$$P_n(t) = P[n events occur in [0,t)]$$

it can be shown that

$$P_n(t) = P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

Proof

Consider the event $[X(t+\triangle t)=n]$; this is equivalent to the union of the disjoint events

$$(x \text{ events occur in } [t,t+\triangle t)) \cap (n-x \text{ events occur in } [0,t)) \text{ for } x=0,1,...,n.$$

Then

$$\begin{aligned}
\mathrm{P}[X(t+\triangle t)=n] &= \sum_{x=0}^{n} \mathrm{P}[X(t+\triangle t)=n|X(t)=n-x]\ \mathrm{P}[X(t)=n-x] \\
&= \sum_{x=0}^{n} \mathrm{P}[X(t+\triangle t)-X(t)=x]\ \mathrm{P}[X(t)=n-x] \qquad \text{(by 2.)} \\
&= \mathrm{P}[X(t+\triangle t)-X(t)=0]\ \mathrm{P}[X(t)=n]+ \\
&\quad\ \mathrm{P}[X(t+\triangle t)-X(t)=1]\ \mathrm{P}[X(t)=n-1]\ +\ O(\triangle t) \qquad \text{(by 4.)}
\end{aligned}$$

so therefore

$$P_n(t+\triangle t) = P_0(\triangle t)\ P_n(t)\ +\ P_1(\triangle t)\ P_{n-1}(t)\ +\ O(\triangle t)$$

Now, direct from assumption 3., $P_1(\triangle t)=\lambda\triangle t + O(\triangle t)$, and therefore

$$P_0(\triangle t) = 1-\lambda\triangle t - O(\triangle t),$$

as $P_0(\triangle t)+P_1(\triangle t)=O(\triangle t)$, so by substitution,

$$P_n(t+\triangle t) = [1-\lambda\triangle t - O(\triangle t)]\ P_n(t)\ +\ [\lambda\triangle t + O(\triangle t)]\ P_{n-1}(t)\ +\ O(\triangle t).$$

Thus, by rearrangement, and taking the limit as $\triangle t \longrightarrow 0$,

$$\begin{aligned}
\frac{dP_n(t)}{dt} &= \lim_{\triangle t \longrightarrow 0}\left[\frac{P_n(t+\triangle t)-P_n(t)}{\triangle t}\right] \\
&= \lambda\left[P_{n-1}(t)-P_n(t)\right]
\end{aligned} \tag{1}$$

For $n=0$,

$$P_0(t+\triangle t)\ =\ P_0(\triangle t)P_0(t)\ =\ [1-\lambda\triangle t - O(\triangle t)]\ P_0(t)$$

and so

$$\begin{aligned}
\frac{dP_0(t)}{dt} &= \lim_{\triangle t \longrightarrow 0}\left[\frac{P_0(t+\triangle t)-P_0(t)}{\triangle t}\right] \\
&= -\lambda P_0(t) \\
\Longrightarrow P_0(t) &= e^{-\lambda t}.
\end{aligned} \tag{2}$$

Substituting into (1) with $n=1$,

$$\frac{dP_1(t)}{dt} \;=\; \lambda\,[P_0(t) - P_1(t)] \;=\; \lambda\left[e^{-\lambda t} - P_1(t)\right]$$

and the solution of this differential equation is easily seen to be

$$P_1(t) \;=\; (\lambda t)\; e^{-\lambda t}.$$

In general,

$$P_n(t) = P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

From previous calculations, it is easily seen that the expected value of $X(t)$ is given by

$$\mathrm{E}[X(t)] = \lambda t$$

so the constant of proportionality, λ, can be interpreted as the **rate of occurrence** or **intensity** of the **Poisson process**, $X(t)$.

The Poisson process is used to model the occurrence of events that are independent in continuous time

Examples : Failures/breakdowns of mechanical components, occurrence of accidents, emission of particles from radioactive sources etc.