7. TRANSFORMATIONS OF RANDOM VARIABLES i.e Continuous r.v. X

Given X with pdf f_X, can we compute the distribution of r.v.

$$Y = g(X)$$

where g is some "transformation" function

- seen scale

location/scale

transforms previously.

pdf f_X

cdf F_X

New r.v. $Y = aX + b$ $\quad a > 0$

$$F_Y(y) = P[Y \le y]$$
$$= P[aX + b \le y]$$
$$= P\left[X \le \left(\frac{y-b}{a}\right)\right] \quad (a>0)$$
$$= F_X\left(\frac{y-b}{a}\right) \quad (a>0)$$

EXAMPLE

The maximum temperature in degrees Fahrenheit, X, measured in a type of chemical reaction varies between experiments according to a pdf f_X given by

$$f_X(x) = x \exp\left\{-\frac{1}{2}x^2\right\} \qquad x > 0$$

and zero otherwise. The maximum temperature measured in degrees Celsius, is a continuous random variable Y defined in terms of X by

$$Y = \frac{5}{9}(X - 32)$$

Note first that the range of the transformed variable is $\mathbb{Y} = \{y : y > -\frac{5}{9}32\}$ and the pdf of Y is computed by first inspecting the cdf of Y. From first principles,

$$F_Y(y) = P[Y \le y] = P\left[\frac{5}{9}(X-32) \le y\right]$$
$$= P\left[X \le \frac{9}{5}y + 32\right]$$
$$= F_X\left(\frac{9}{5}y + 32\right)$$

On differentiation using the chain rule, and recalling that the derivative of the cdf is the pdf, we have

$$f_Y(y) = \frac{9}{5}f_X\left(\frac{9}{5}y + 32\right) = \frac{9}{5}\left(\frac{9}{5}y + 32\right)\exp\left\{-\frac{1}{2}\left(\frac{9}{5}y + 32\right)^2\right\} \qquad y > -\frac{5}{9}32$$

$\Rightarrow$ by differentiating

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(must remember to define $\mathbb{Y}$)

Now $\quad Y = aX + b \Leftrightarrow X = \left(\frac{Y-b}{a}\right)$

Is it good enough to just substitute

$$\frac{y-b}{a}$$

into the density function for f_X.

EXAMPLE

Molecule speed X with pdf

$$f_X(x) = \alpha x^2 e^{-\beta x^2} \quad x > 0$$

for $\alpha, \beta > 0$.

Kinetic Energy $Y = \frac{m}{2} X^2$

What is the pdf of Y ?

– could plug in $y = \frac{m}{2} x^2$ to f_X

In general this gives the wrong answer.

How do we proceed in general ?

If g is 1-1, so that g^{-1} is a well defined function

$$F_Y(y) = P[Y \leq y]$$

$$= P[g(X) \leq y]$$

$$= P[X \leq g^{-1}(y)] \quad (\mathrm{I})$$

or

$$= P[X \geq g^{-1}(y)] \quad (\mathrm{II})$$

EXAMPLE

If continuous random variable U has a Uniform distribution on the interval $(0,1)$, consider the random variable X defined by

$$X = 1 + \left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor$$

for parameter θ $(0 < \theta < 1)$, where $\lfloor a \rfloor$ is the integer part of a for real value a. The range of the transformed variable is the set

$$\mathbb{X} \equiv \{1, 2, 3, \ldots\}$$

and from first principles, for $x \in \mathbb{X}$

$$\begin{aligned} F_X(x) &= \mathrm{P}[X \leq x] = \mathrm{P}\left[1 + \left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor \leq x\right] \\ &= \mathrm{P}\left[\left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor \leq x - 1\right] \\ &= \mathrm{P}\left[\left\lfloor \frac{\log U}{\log(1-\theta)} \right\rfloor < x\right] && \text{as } x \text{ is integer-valued} \\ &= \mathrm{P}[\log U > x \log(1-\theta)] && \text{as } 0 < \theta < 1 \implies \log(1-\theta) < 0 \\ &= \mathrm{P}[U > (1-\theta)^x] = 1 - F_U((1-\theta)^x) \\ &= 1 - (1-\theta)^x && x = 1, 2, 3, \ldots \end{aligned}$$

and hence

$$X \sim Geometric(\theta)$$

I) g is increasing

e.g. $g(X) = e^X$

$g(X) = \dfrac{e^X}{1 + e^X}$ etc

II) g is decreasing

e.g. $g(X) = e^{-X}$

$g(X) = \dfrac{1}{1 + X}$ etc.

EXAMPLE

If continuous random variable U has a Uniform distribution on the interval $(0,1)$, so that

$$f_U(u) = 1 \qquad F_U(u) = u \qquad 0 < u < 1$$

then to find the probability distribution of random variable X defined by

$$X = \frac{1}{\lambda}\log\left(\frac{U}{1-U}\right)$$

we proceed as follows: By inspection, the range of the transformed variable is $(-\infty, \infty)$, and from first principles,

$$\begin{aligned} F_X(x) &= \mathrm{P}[X \le x] \\ &= \mathrm{P}\left[\frac{1}{\lambda}\log\left(\frac{U}{1-U}\right) \le x\right] \\ &= \mathrm{P}\left[U \le \frac{e^{\lambda x}}{1+e^{\lambda x}}\right] \\ &= F_U\left(\frac{e^{\lambda x}}{1+e^{\lambda x}}\right) = \frac{e^{\lambda x}}{1+e^{\lambda x}} \end{aligned}$$

and hence on differentiation, we have

$$f_U(u) = \frac{(1+e^{\lambda x})\lambda e^{\lambda x} - e^{\lambda x}\lambda e^{\lambda x}}{(1+e^{\lambda x})^2} = \frac{\lambda e^{\lambda x}}{(1+e^{\lambda x})^2} \qquad x \in \mathbb{R}$$

If g is <u>not</u> 1-1

e.g. $g(X) = X^2$

$g(X) = \sin X$

etc

We have still

$$F_Y(y) = P[Y \le y] = P[g(X) \le y]$$

but <u>must</u> <u>take</u> <u>care</u> from now on . . .

EXAMPLE

$X \sim \text{Normal}(0,1) \quad (F_X \equiv \Phi)$

$Y = X^2 \quad (f_X = \phi)$

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[X^2 \le y] \\ &= P[-\sqrt{y} \le X \le \sqrt{y}] \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \\ &= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) \end{aligned}$$

$$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}}\left[\phi(\sqrt{y}) + \phi(-\sqrt{y})\right]$$

$$= y^{1/2-1}\left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}y\right\}$$

$$\equiv \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) \text{ pdf}$$

$$\equiv \chi^2_1$$

(CHI-SQUARED(1))

as X is a positive random variable. Hence the pdf is obtained by differentiation as

$$\begin{aligned} f_Y(y) &= \sqrt{\frac{2}{m}}\,\frac{1}{2}\frac{1}{\sqrt{y}}\,f_X\left(\sqrt{\frac{2y}{m}}\right) = \sqrt{\frac{1}{2my}}\,4\sqrt{\frac{\lambda^3}{\pi}}\left(\sqrt{\frac{2y}{m}}\right)^2 \exp\left\{-\lambda\left(\sqrt{\frac{2y}{m}}\right)^2\right\} \\ &= 4\sqrt{\frac{2\lambda^3}{\pi m^3}}\,y^{1/2}\exp\left\{-\frac{2\lambda}{m}y\right\} \end{aligned}$$

That is, we have that, inspecting the terms in y

$$Y \sim Gamma\left(\frac{3}{2}, \frac{2\lambda}{m}\right)$$

Note here that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ and

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{1/2-1}e^{-x}\,dx = \int_0^\infty (t^2)^{-1/2}e^{-t^2}(2t)\,dt = 2\int_0^\infty e^{-t^2}\,dt = \int_{-\infty}^\infty e^{-t^2}\,dt = \sqrt{\pi}$$

setting $t^2 = x$, and using the integral result from the Normal pdf proof.

EXAMPLE

Random variable X measures the speed of a molecule of mass m in a gas at some temperature. Kinetic theory suggests that the pdf of X can be expressed as

$$f_X(x) = 4\sqrt{\frac{\lambda^3}{\pi}}\,x^2\exp\left\{-\lambda x^2\right\} \qquad x > 0$$

for some constant $\lambda > 0$. The kinetic energy of the molecule is a continuous random variable Y defined by

$$Y = \frac{mX^2}{2}$$

The pdf of Y is computed as follows from the cdf; for $y > 0$

$$\begin{aligned} F_Y(y) &= \mathrm{P}[Y \le y] = \mathrm{P}\left[\frac{mX^2}{2} \le y\right] \\ &= \mathrm{P}\left[X^2 \le \frac{2y}{m}\right] \\ &= \mathrm{P}\left[-\sqrt{\frac{2y}{m}} \le X \le \sqrt{\frac{2y}{m}}\right] \\ &= \mathrm{P}\left[X \le \sqrt{\frac{2y}{m}}\right] - \mathrm{P}\left[X < -\sqrt{\frac{2y}{m}}\right] = F_X\left(\sqrt{\frac{2y}{m}}\right) \end{aligned}$$

(ii) If T is constant, but V has density

$$f_V(v) = \frac{4}{\sqrt{\pi}}v^2\exp\left\{-v^2\right\} \qquad 0 < v$$

then X has range $(0, \infty)$, and

$$F_X(x) = \mathrm{P}[X \le x] = \mathrm{P}\left[\frac{V^2}{g}\sin 2T \le x\right] = \mathrm{P}\left[V^2 \le \frac{gx}{\sin 2T}\right] = \mathrm{P}\left[V \le \sqrt{\frac{gx}{\sin 2T}}\right] = F_V\left(\sqrt{\frac{gx}{\sin 2T}}\right)$$

and on differentiation, we have

$$f_X(x) = \frac{4}{\sqrt{\pi}}\frac{gx}{\sin 2T}\exp\left\{-\frac{gx}{\sin 2T}\right\}\cdot\sqrt{\frac{g}{\sin 2T}}\,\frac{1}{2}\frac{1}{\sqrt{x}} = \sqrt{\frac{4x}{\pi}}\sqrt[3]{\frac{g}{\sin 2T}}\exp\left\{-\frac{gx}{\sin 2T}\right\} \qquad 0 < x$$

Again

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp\left\{-\beta x\right\}$$

where

$$\alpha = \frac{3}{2} \qquad \beta = \frac{g}{\sin 2T}$$

as again $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$, and hence

$$X \sim Gamma\left(\frac{3}{2}, \frac{g}{\sin 2T}\right).$$

EXAMPLE

A projectile is fired from the origin at velocity V and angle T from the horizontal. It lands a distance X away, where for gravitational constant g,

$$X = \frac{V^2}{g}\sin 2T$$

(i) If V is constant, but T has a Uniform distribution on $(0, \pi/2)$ then

$$f_T(t) = \frac{2}{\pi} \quad 0 < t < \frac{\pi}{2} \qquad F_T(t) = \frac{2t}{\pi} \quad 0 < t < \frac{\pi}{2}$$

The range of X is $\left(0, \frac{V^2}{g}\right)$, and for x in this range, the cdf of X is obtained as follows: by definition,

$$\mathrm{P}[X \le x] = \mathrm{P}\left[\frac{V^2}{g}\sin 2T \le x\right]$$

Hence, by inspection of the graph for sine, and consideration of the inverse sine function, we have that

$$\frac{V^2}{g}\sin 2T \le x \iff 2T \le \sin^{-1}\left(\frac{gx}{V^2}\right) \quad \text{or} \quad 2T \ge \pi - \sin^{-1}\left(\frac{gx}{V^2}\right)$$

as the sin function is not 1-1. Hence

$$\mathrm{P}[X \le x] = \mathrm{P}\left[2T \le \sin^{-1}\left(\frac{gx}{V^2}\right)\right] + \mathrm{P}\left[2T \ge \pi - \sin^{-1}\left(\frac{gx}{V^2}\right)\right]$$

and so

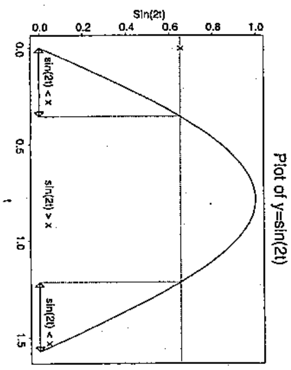
$$F_X(x) = F_T\left(\frac{1}{2}\sin^{-1}\left(\frac{gx}{V^2}\right)\right) + 1 - F_T\left(\frac{1}{2}\left(\pi - \sin^{-1}\left(\frac{gx}{V^2}\right)\right)\right)$$

which simplifies to

$$F_X(x) = \frac{2}{\pi}\frac{1}{2}\sin^{-1}\left(\frac{gx}{V^2}\right) + 1 - \frac{2}{\pi}\frac{1}{2}\left(\pi - \sin^{-1}\left(\frac{gx}{V^2}\right)\right) = \frac{2}{\pi}\sin^{-1}\left(\frac{gx}{V^2}\right) \qquad 0 < x < \frac{V^2}{g}$$

On differentiation, we have the density of X as

$$f_X(x) = \frac{2}{\pi\sqrt{V^4/g^2 - x^2}} \qquad 0 < x < \frac{V^2}{g}$$



If X is <u>discrete</u>, look at p.m.f. of Y. $\quad$ If X is <u>continuous</u>, look at cdf

$$f_Y(y) = P[Y = y] = P[g(X) = y] = P[X \in A_y]$$

where $A_y = \{x \in \mathcal{X} : g(x) = y\}$

– i.e. sum the probabilities of all points x that map to y under g.

$$F_Y(y) = P[Y \le y] = P[g(X) \le y] = P[X \in B_y] = \int_{B_y} f_X(x)\,dx$$

where $B_y = \{x \in \mathcal{X} : g(x) \le y\}$

If g is 1-1 [MONOTONE]

(g^{-1} is uniquely defined)

DISCRETE CASE

$$f_Y(y) = f_X(g^{-1}(y)) \quad y \in \mathcal{Y}$$

CONTINUOUS CASE

$$f_Y(y) = f_X(g^{-1}(y)) \quad J(y)$$

where

$$J(y) = \left| \frac{d}{dt}\left\{ g^{-1}(t) \right\}_{t=y} \right|$$

is the JACOBIAN of the transform".

(see handout, p 5,6)