Chapter 6

6. Continuous Random Variables

So far, we have only defined random variables and their distributions in the discrete case

i.e $\Omega \equiv \{\omega_1, \omega_2, \dots\}$

$\rightarrow \mathbb{X} \equiv \{x_1, x_2, \dots\}$

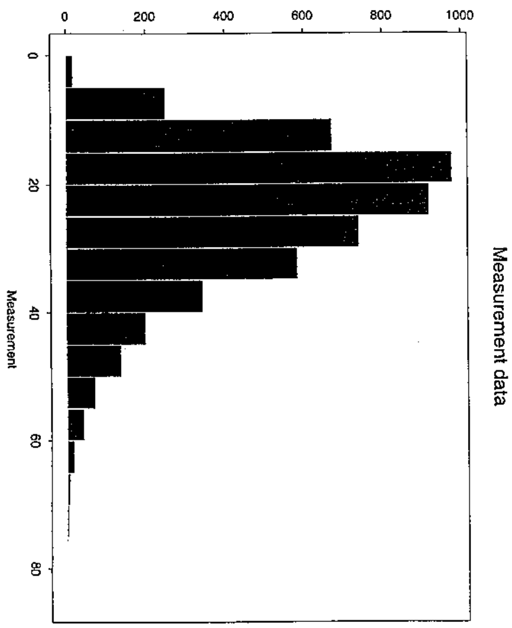
where $\mathbb{X}$ is COUNTABLE

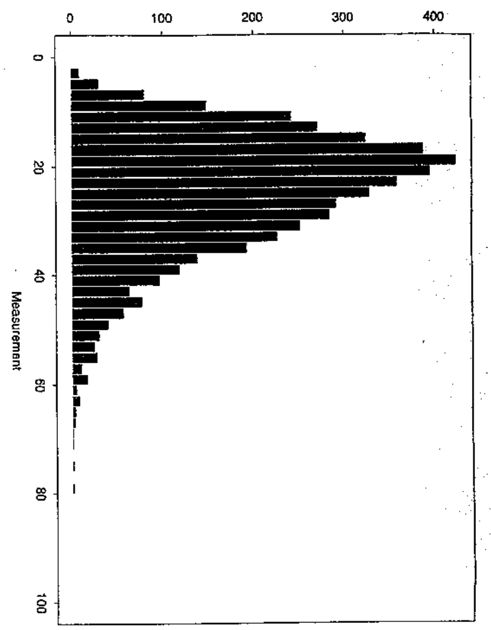
However, experimental considerations also require us to model UNCOUNTABLE sample spaces

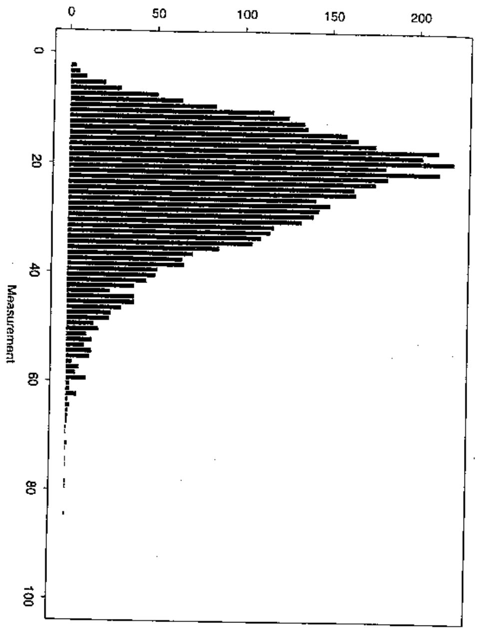
e.g in any "MEASUREMENT" experiment

- height, weight
- time
- temperature etc

We believe that there are a continuum of outcomes; measurement can take any real value in some range.







LEAF LENGTH DATA.

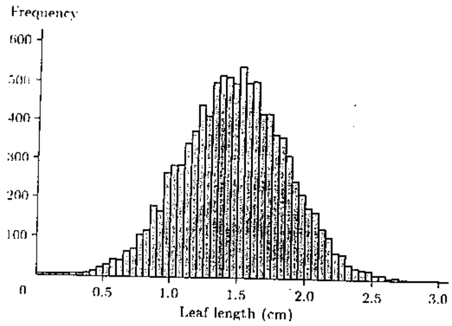
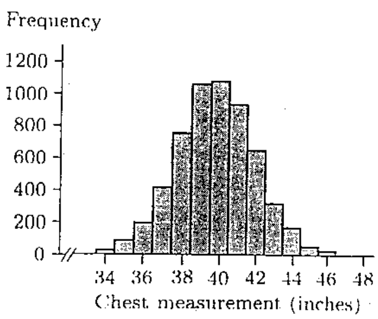


Figure 2.9 A histogram based on a very large sample



Chest measurements of 5732 Scottish soldiers

The discrete framework cannot deal with such experiments.

However, we can still consider a random variable definition

$$X : \Omega \longrightarrow \mathbb{R}$$
$$\omega \longmapsto x$$

in the continuous context, where Ω is uncountable.
We merely must be more careful in out probability specifications.

It is still sensible to consider, for example,

$$P[X \leq x]$$

in the continuous domain; a little harder to consider

$$P[X = x]$$

[consider histogram with "bin" widths that decrease to zero – counts also tend to zero!]

The range of X, denoted $\mathbb{X}$ could be

- a bounded interval in $\mathbb{R}$
- the union of bounded intervals
- $\mathbb{R}^+ \equiv \{x \in \mathbb{R} : x > 0\}$
- $\mathbb{R}$

6.1 Continuous Random Variables

Definition

A map X from sample space Ω to $\mathbb{R}$

$$X : \Omega \longrightarrow \mathbb{R}$$
$$\omega \longmapsto x$$

so that $X(\omega) = x$.

is a continuous random variable if Ω is uncountable and if

$$P[X \leq x] \equiv P(A_x)$$

is a continuous function of x

$$A_x \equiv \{\omega \in \Omega : X(\omega) \leq x\}$$

6.2 Continuous Cumulative Distribution Function

Definition

If X is a continuous r.v., then the continuous cumulative distribution function (cdf) is denoted F_X and defined by

$$F_X(x) = P[X \leq x]$$

for $x \in \mathbb{R}$

This definition (and notation) is identical to the discrete case.

In fact, we require that the continuous cdf obeys very similar rules to the discrete cdf.

We require that

(i) F_X is non-decreasing

(ii) F_X is continuous

(iii) $\lim_{x \to -\infty} F_X(x) = 0$

(iv) $\lim_{x \to \infty} F_X(x) = 1$.

6.3 PROBABILITY DENSITY FUNCTION

DEFINITION

If X is a continuous r.v. with cdf F_X, then the probability density function, or pdf, is denoted f_X and is defined (implicitly) by

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

i.e

$$f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x}$$

wherever F_X is differentiable.

(NOTE : RHS is merely the first derivative of F_X with respect to x)

The continuous cdf F_X
and pdf f_X

specify a probability model

For formal probability statements we must use

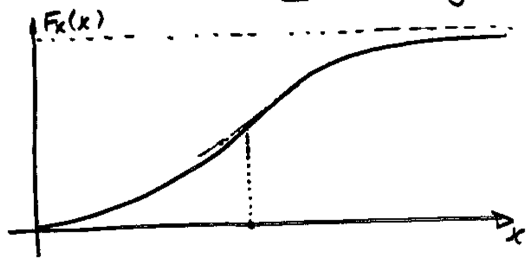
$$F_X(x) = P[X \leq x]$$

but often the pdf f_X is a more convenient route.

Recall that

$0 \leq F_X(x) \leq 1$

and that $F_X(x)$ is non-decreasing

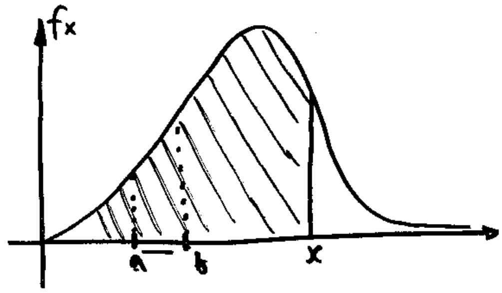


Note that $f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x}$

must be positive

To calculate F_X from f_X

– INTEGRATE UP TO x



i.e $P[X \leq x]$ is the integral under f_X up to x

By construction, for real a, b

$$P[a < X \leq b] = P[X \leq b] - P[X \leq a]$$

as

$$(-\infty, b] \equiv (-\infty, a] \cup (a, b]$$

$\therefore$ by PROB AXIOM III

$$P[X \in (-\infty, b]] = P[X \in (-\infty, a]] + P[X \in (a, b]]$$

etc.

$$\therefore P[a < X \leq b] = F_X(b) - F_X(a)$$

PROPERTIES OF f_X

We require

(i) $f_X(x) \geq 0 \quad x \in \mathbb{R}$

(ii) $\int_{\mathbb{X}} f_X(x)\, dx = 1$

[by convention $f_X(x) = 0$, $x \notin \mathbb{X}$]

NOTE In the continuous case

We do not require

$$0 \le f_X(x) \le 1$$

but merely that

$$0 \le f_X(x)$$

as f_X is a "rate of change" of probability, not a probability.

NOTE

"F_X is continuous at x_0"

$\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ s.t.

$$|F_X(x) - F_X(x_0)| < \varepsilon$$

whenever

$$|x - x_0| < \delta$$

But consider, for some $h > 0$

$$P[x_0 < X \le x_0 + h]$$

$$= F_X(x_0 + h) - F_X(x_0)$$

"Continuity" $\Rightarrow$ as $h \to 0$

$$P[X = x_0] = F_X(x_0) - F_X(x_0)$$

$$= 0$$

$\therefore$ The pdf f_X is a useful modelling facility

BUT IT DOES NOT SPECIFY PROBABILITIES!

How to choose f_X:

(i) to model data histogram shapes?

(ii) in any way such that the two requirements are met.

CONSTRUCTING PDFs

Recall we require

(i) $f_X(x) \geq 0$

(ii) $\int_{\mathbb{X}} f_X(x)\,dx = 1$.

Take <u>any</u> function f that is

(i) non-negative

(ii) integrable on $\mathbb{X}$

Suppose

$$\int_{\mathbb{X}} f(x)\,dx = c < \infty$$

Then can define a pdf by

$$f_X(x) = \frac{1}{c} f(x)$$

- two requirements for f_X are met by construction.

NOTE

All pdfs can be thought of as

constant $\times$ function of x

The constant is termed the "normalizing" constant.

We can again look at

EXPECTATION

and

GENERATING FUNCTIONS

in the continuous case.

For expectation, we retain the idea of a "centre of probability mass" from the discrete case, but replace summation by integration.

DEFINITION

If X is a continuous r.v. with range $\mathbb{X}$ and pdf f_X, the expectation of X is defined by

$$E_{f_X}[X] = \int_{\mathbb{X}} x f_X(x)\,dx$$

whenever this integral is convergent)

The expectation of X is a real constant describing the "centre of density" or location" of the pdf on the x-axis.

– a weighted average of values in $\mathbb{X}$ with weights given by f_X

NOTE

$E_{f_X}[X]$ need not be finite!

A GENERATING FUNCTION FOR CONTINUOUS R.Vs

Recall the p.g.f. for discrete variables

$$G_X(t) = \sum_{x \in \mathbb{X}} f_X(x) t^x$$

i.e. the generating function for the probabilities

For continuous variables, we may replace the sum by an integral to get

$$G_X(t) = \int_{\mathbb{X}} f_X(x) t^x \,dx$$

i.e. G_X is a function of $t \in \mathbb{R}$

This is not a generating function for probabilities

But note that

$$G_X(1) = 1$$

and also that if

$$G_X^{(r)}(1) = \frac{d^r}{dt^r}\Big\{ G_X(t) \Big\}_{t=1}$$

then, by differentiating under the integral

$$G_X^{(r)}(1) = \int_{\mathbb{X}} x(x-1)\dots(x-r+1)\, f_X(x)\,dx$$

$$\therefore \quad G_X^{(1)}(1) = \int_{\mathbb{X}} x f_X(x)\,dx = E_{f_X}[X]$$

as

$$\frac{d^r}{dt^r}\Big\{G_X(t)\Big\} = \int_{\mathbb{X}} f_X(x) \frac{d^r}{dt^r}\{t^x\}\,dx$$

NOTE This result also holds for discrete variables

E.G. $X \sim \text{Geometric}(\theta)$

$$G_X(t) = \frac{\theta t}{1-(1-\theta)t}$$

$$\Rightarrow G_X^{(1)}(t) = \frac{[1-(1-\theta)t]\theta + \theta t(1-\theta)}{[1-(1-\theta)t]^2}$$

$$\therefore E_{f_X}[X] = G_X^{(1)}(1) = \frac{1}{\theta}$$

EXAMPLE Random variable X

Range $\mathbb{X} \equiv \mathbb{R}^+ \equiv \{x : x > 0\}$

PDF

$$f_X(x) = \frac{c}{(1+x)^{\alpha+1}} \qquad x > 0.$$

for parameter $\alpha > 0$.

(i.) calculate c

- need $\int_{\mathbb{X}} f_X(x)\,dx = 1$

$\therefore$ Need

$$\int_0^\infty \frac{c}{(1+x)^{\alpha+1}}\,dx = 1$$

$$\therefore \quad c^{-1} = \int_0^\infty \frac{1}{(1+x)^{\alpha+1}}\,dx$$

$$= \left[\frac{-1}{\alpha} \frac{1}{(1+x)^\alpha} \right]_0^\infty \quad (\text{as } \alpha > 0)$$

$$= \frac{1}{\alpha}$$

$$\Rightarrow \quad c = \alpha$$

$$\therefore \quad f_X(x) = \frac{\alpha}{(1+x)^{\alpha+1}} \qquad x>0$$

CDF For $x>0$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \int_0^x \frac{\alpha}{(1+t)^{\alpha+1}}\,dt$$

$$= \left[\frac{-1}{(1+t)^{\alpha}} \right]_0^x$$

$$= 1 - \frac{1}{(1+x)^{\alpha}} \qquad x>0$$

EXPECTATION

$$E_{f_X}[X] = \int_{\mathbb{X}} x f_X(x)\,dx$$

$$= \int_0^{\infty} x \cdot \frac{\alpha}{(1+x)^{\alpha+1}}\,dx$$

$$= \begin{cases} \dfrac{1}{\alpha-1} & \alpha>1 \\ \infty & \alpha<1 \end{cases}$$

(INTEGRATION BY PARTS)

EXAMPLE $\mathbb{X} \equiv (0, \theta)$

$$f_X(x) = \frac{1}{\theta} \qquad 0<x<\theta$$

for parameter $\theta>0$.

CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

$$= \int_0^x \frac{1}{\theta}\,dt$$

$$= \frac{x}{\theta} \qquad 0<x<\theta$$

EXPECTATION

$$E_{f_X}[X] = \int_{\mathbb{X}} x f_X(x)\,dx$$

$$= \int_0^{\theta} x \cdot \frac{1}{\theta}\,dx = \frac{\theta}{2}.$$

GENERATING FUNCTION

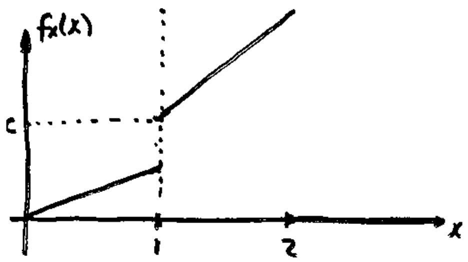
$$G_X(t) = \int_{\mathbb{X}} f_X(x)\,t^x\,dx$$

$$= \int_0^{\theta} \frac{1}{\theta}\,t^x\,dx$$

$$= \left[\frac{t^x}{\theta \log t} \right]_0^{\theta} = \frac{t^{\theta}-1}{\theta \log t}$$

EXAMPLE $\mathbb{X} \equiv (0,2]$

$$f_X(x) = \begin{cases} \frac{c}{2}x & 0 < x \leq 1 \\ cx & 1 < x \leq 2 \end{cases}$$



PDF discontinuous ?

— OK, we only require CDF continuous.

CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

Now, if $0 < x \leq 1$

$$F_X(x) = \int_0^x \frac{c}{2} t \, dt$$

$$= \left[\frac{c}{2}\frac{t^2}{2}\right]_0^x = \frac{cx^2}{4}$$

But if $1 \leq x \leq 2$

$$F_X(x) = \int_0^x f_X(t)\,dt$$

$$= \int_0^1 f_X(t)\,dt + \int_1^x f_X(t)\,dt$$

$$= F_X(1) + \int_1^x ct\,dt$$

$$= \frac{c}{4} + \left[\frac{ct^2}{2}\right]_1^x$$

$$= \frac{c}{4} + \frac{c}{2}\left[x^2 - 1\right]$$

$$= \frac{c}{4}\left[2x^2 - 1\right]$$

At $x = 2$

$$F_X(2) = 1 \Rightarrow c = \frac{4}{7}$$

$$\therefore f_X(x) = \begin{cases} \frac{2}{7}x & 0 < x \leq 1 \\ \frac{4}{7}x & 1 < x \leq 2 \end{cases}$$

$$F_X(x) = \begin{cases} \frac{x^2}{7} & 0 < x \leq 1 \\ \frac{2x^2 - 1}{7} & 1 < x \leq 2 \end{cases}$$

i.e ranges are $(0,1]$ and $(1,2]$

NOTE In the continuous domain

$$P[X = x] = 0 \qquad \forall x$$

$\therefore$ It follows that, for $a < b$

$$P[a < X \leq b]$$
$$= P[a < X < b]$$
$$= P[a \leq X < b]$$
$$= P[a \leq X \leq b]$$

We now seek continuous probability models for typical experimental problems.

- often, the range $\mathbb{X}$ indicates which models are appropriate

We consider

(1) $\mathbb{X} \equiv \mathbb{R}$

(2) $\mathbb{X} \equiv \mathbb{R}^+$

- reliability
  survival analysis
  actuarial modelling

(3) Bounded intervals (a, b)

6.4 THE UNIFORM DISTRIBUTION

Suppose $\mathbb{X} \equiv (a, b)$

$\mathcal{E}$ - "all points equally likely"

or equivalently

"all intervals of the same length within $\mathbb{X}$ have the same probability content"

i.e. prob content of

$$(x, x+\delta) \subset \mathbb{X}$$

is proportional to δ

$\therefore$ f_X is constant on $\mathbb{X}$

and

$$f_X(x) = \frac{1}{b-a} \qquad x \in (a, b)$$

and zero otherwise

(as we require $\int_a^b f_X(x)\,dx = 1$)

This is the continuous Uniform dist^n.

$$X \sim \text{Uniform}(a,b)$$

$$F_X(x) = \frac{x-a}{b-a} \qquad a<x<b$$

$$E_{f_X}[X] = \frac{a+b}{2}$$

SPECIAL CASE $\quad a=0, b=1$

$$f_X(x) = 1 \qquad 0<x<1$$

- the STANDARD UNIFORM.

EXTENSION 2D models

"Select point at random from within unit disc D"

$\mathcal{E}$ - "all points in D equally likely"

To formulate a probability model, need a 2D representation for (x,y) coordinates jointly.

$\longrightarrow$ JOINT PROBABILITY DISTRIBUTIONS.

6.5 THE EXPONENTIAL DISTRIBUTION

$$\mathbb{X} \equiv \mathbb{R}^+$$

- need an integrable, non-negative function on $\mathbb{X}$

$$f(x) = e^{-x} \qquad x>0$$

or

$$f(x) = e^{-\lambda x} \qquad x>0$$

for parameter $\lambda>0$.

$$\int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda}$$

$\therefore$ Define

$$f_X(x) = \lambda e^{-\lambda x} \qquad x>0$$

for $\lambda>0$.

This is the EXPONENTIAL DISTRIBUTION

CDF $\quad F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$$= \int_0^x \lambda e^{-\lambda t}\,dt$$

$$= 1-e^{-\lambda x} \qquad x>0$$

Note

Define the **Reliability Function** R_X by

$$R_X(x) = P[X > x] = 1 - P[X \le x]$$
$$= 1 - F_X(x)$$

For the Exponential dist^n

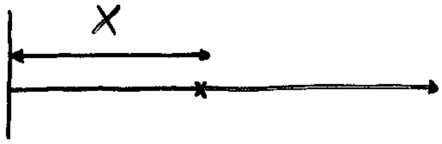
$$R_X(x) = e^{-\lambda x}$$

- sometimes R_X is easier to construct.

Interpretation

The Exponential is the simplest continuous "waiting-time" distribution

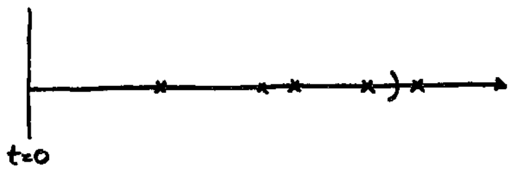
- it is analogous to the Geometric in the discrete case.
- used to model survival / failure time



Typically we assume $X \sim Exp(\lambda)$

λ is the rate parameter.

Recall the Poisson Process



X_t = "number of events in $[0,t)$"

Poisson Process Assumptions

$$\Rightarrow X_t \sim Poisson(\lambda t)$$
$$X(t) \equiv X_t.$$

Let

T - "time to first event"

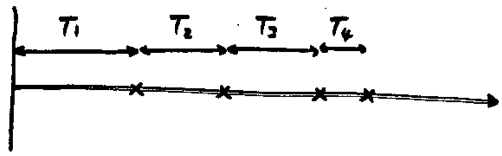
(T is a cts r.v. with range $\mathbb{R}^+$)

Then

$$F_T(t) = P[T \le t] \qquad t>0$$
$$= 1 - P[T > t]$$
$$= 1 - P[X(t) = 0]$$
$$= 1 - e^{-\lambda t} \qquad \text{POISSON PROCESS ASSUMPTION}$$

$$\Rightarrow T \sim Exponential(\lambda)$$

Further



T_i - "time from $(i-1)^{st}$ to i^{th} event"

Then Poisson process assumptions
(INDEPENDENCE, CONSTANT RATE)

$\Rightarrow$ $T_1, T_2, T_3, \dots$ are independent and identically distributed (i.i.d.)

$$T_i \sim Exp(\lambda).$$

NOTE We could write the Exponential pdf as

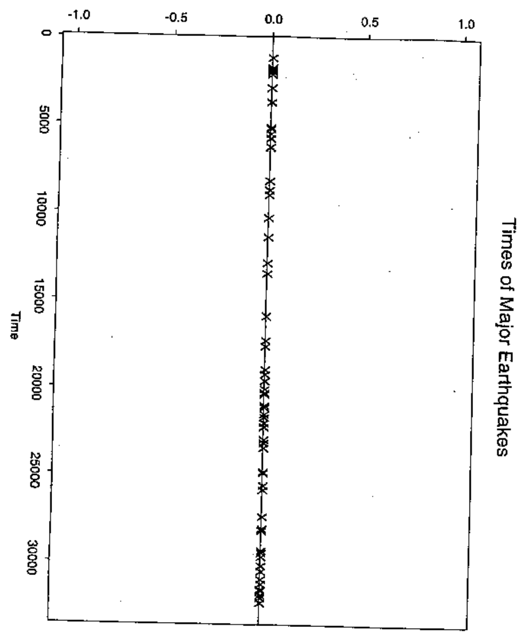
$$f_X(x) = \frac{1}{\mu} e^{-x/\mu} \qquad x > 0$$

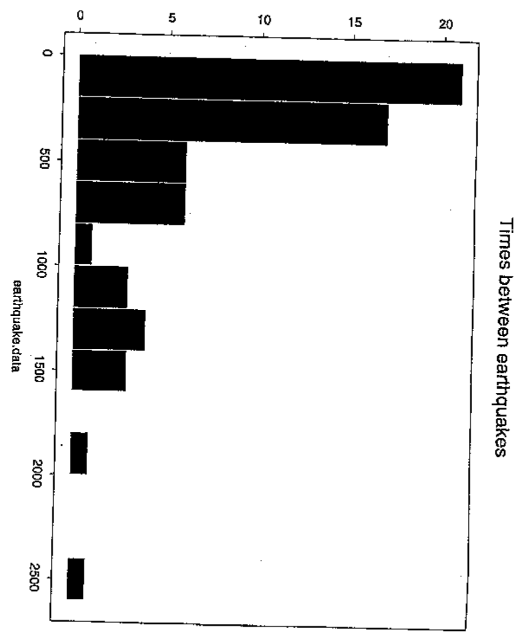
for parameter $\mu > 0$. $\left(\mu = \frac{1}{\lambda}\right)$

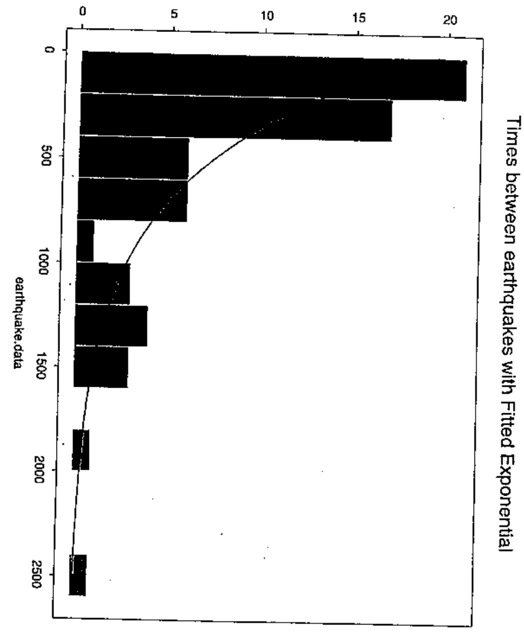
This is the same model "reparameterized"

This parameterization is used because

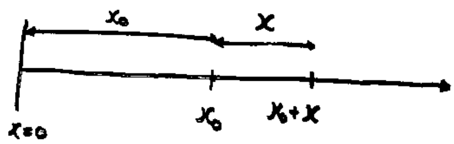
$$E_{f_X}[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx$$
$$= \frac{1}{\lambda} = \mu.$$







THE LACK OF MEMORY PROPERTY



$$P[X > x_0 + x \mid X > x_0]$$

$$= \frac{P[(X > x_0+x) \cap (X > x_0)]}{P[X > x_0]}$$

$$= \frac{P[X > x_0 + x]}{P[X > x_0]} = \frac{e^{-\lambda(x_0+x)}}{e^{-\lambda x_0}}$$

$$= e^{-\lambda x} = P[X > x]$$

– this property is unique to the Exponential dist^{n}.

GENERATING FUNCTION

$$G_X(t) = \int_{\mathbb{X}} f_X(x)\, t^x \, dx$$

$$= \frac{\lambda}{\lambda - \log t} \quad \text{if } \lambda > \log t \quad \text{(natural log)}$$

(CHECK)

NOTE Suppose $X \sim Exp(1)$

Consider r.v. $Y = X/\lambda \quad \lambda > 0$

(e.g. X measured in seconds, Y measured in hours $\Big\} \Rightarrow \lambda = 60^2$)

"RE-SCALING" or "SCALE TRANSFORM"

What is the probability distribution of r.v. Y? Look at cdf:

$$F_Y(y) = P[Y \le y] \quad y > 0$$

$$= P[X/\lambda \le y]$$

$$= P[X \le \lambda y]$$

$$= F_X(\lambda y)$$

But $X \sim Exp(1) \Rightarrow F_X(x) = 1 - e^{-x}$

$$\Rightarrow F_Y(y) = F_X(\lambda y) = 1 - e^{-\lambda y}$$

$$\Rightarrow Y \sim Exp(\lambda)$$

6.6 THE GAMMA DISTRIBUTION

Again consider $\mathbb{X} \equiv \mathbb{R}^+$.

We seek a more general probability model for this range - require

NON-NEGATIVE + INTEGRABLE

functions on $\mathbb{X}$.

Try

$$f(x) = x^n e^{-x}$$

for $n = 0, 1, 2, \ldots$

Non-negative, and

$$I_n = \int_0^\infty x^n e^{-x}\, dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\, dx$$

$$= 0 + n I_{n-1}$$

$$= n(n-1) I_{n-2} \ldots = n!$$

$\therefore$ Integrable, with

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

$\therefore$ Could define f_X by

$$f_X(x) = \frac{1}{n!} x^n e^{-x} \qquad x > 0.$$

<u>But</u>, actually

$$\int_0^\infty x^{\alpha-1} e^{-x}\, dx < \infty$$

for <u>any</u> $\alpha > 0$. (not just integer values)

$\therefore$ Consider

$$f_X(x) = c\, x^{\alpha-1} e^{-x} \qquad x > 0$$

for parameter $\alpha > 0$, where

$$c^{-1} = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

No closed form for c .... but it can be evaluated <u>numerically</u>

(just like "log" or "e" functions)

DEFINITION

For $t>0$, define the GAMMA FUNCTION, $\Gamma(t)$, by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$$

PROPERTIES

(i) By parts $\Gamma(t) = (t-1)\Gamma(t-1)$

(ii) If $t = 1, 2, 3 ..$

$$\Gamma(t) = (t-1)!$$

The pdf f_X

$$f_X(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \quad x>0$$

is the pdf of the Gamma Distribution with parameter α.

CDF : Not available in closed form

EXPECTATION :

$$E_{f_X}[X] = \int_0^\infty x f_X(x)\, dx$$

$$= \int_0^\infty x . \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} d$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{(\alpha+1)-1} e^{-x}\, dx.$$

$$= \frac{1}{\Gamma(\alpha)} . \Gamma(\alpha+1) = \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \alpha$$

Now, we again consider a "scale transformation" to generalize the Gamma pdf.

Let $Y = \frac{X}{\beta}$

Again consider F_Y : for $y>0$

$$F_Y(y) = P[Y \le y]$$

$$= P\left[\frac{X}{\beta} \le y\right] = P[X \le \beta y]$$

$$= F_X(\beta y)$$

$$\Rightarrow f_Y(y) = \beta f_X(\beta y) \quad \text{(DIFFERENTIATI}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} \quad y>0$$

This is the general form of the Gamma pdf

$$Y \sim Gamma(\alpha, \beta)$$

NOTE

$$E_{f_Y}[Y] = \int_0^\infty y \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}$$

$$= \frac{\alpha}{\beta}.$$

NOTE We have proved that

$$\int_0^\infty y^{\alpha-1} e^{-\beta y} dy = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

- can use this result in subsequent calcs.

α is the SHAPE parameter

β is the SCALE parameter

If $Y = X/\beta$ then

$$E_{f_Y}[Y] = \frac{1}{\beta} E_{f_X}[X]$$

Connection to other distributions

Recall the Poisson process with rate λ

if X_i = "time interval between $(i-1)^{st}$ and i^{th} event"

saw that

$$X_1, X_2, \ldots \sim Exponential(\lambda)$$

If Y_n = "time until n^{th} event"

ASSESSED 3 Q2

$$\Rightarrow f_{Y_n}(y) = \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}$$

i.e $Y_n \sim Gamma(n, \lambda)$

NOTE ALSO : $Y_n = \sum_{i=1}^{n} X_i$

SPECIAL CASE OF GAMMA DIST^N

Suppose, for $n = 1, 2, 3, \ldots$

$$X \sim Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

(i.e. $\alpha = \frac{n}{2}$, $\beta = \frac{1}{2}$)

Then X has a Chi-Squared dist^n with n degrees of freedom.

$$X \sim \chi^2_n.$$

6.7 THE NORMAL DISTRIBUTION

Suppose $\mathbb{X} \equiv \mathbb{R}$

We seek a non-negative / integrable function on $\mathbb{R}$.

Try $$f(x) = \exp\{-x^2\}$$

- positive
- integrable
- symmetric about 0.

Can prove

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{\infty} \exp\{-x^2\}\, dx = \sqrt{\pi}$$

(see handout)

+ deduce

$$\int_{-\infty}^{\infty} \exp\{-\lambda x^2\}\, dx = \sqrt{\frac{\pi}{\lambda}}$$

for $\lambda > 0$.

(change variables to $t = \dfrac{x}{\sqrt{\lambda}}$ in integral)

First, we compute the integral of the function $f(x)$ Suppose that, say,

$$\int_{-\infty}^{\infty} \exp\{-x^2\}\, dx = c$$

(we know by inspection that f is integrable, and $c > 0$). Then

$$\begin{aligned} c &= \int_{-\infty}^{\infty} \exp\{-x^2\}\, dx \\ &= \frac{1}{c}\int_{-\infty}^{\infty} c\exp\{-x^2\}\, dx \\ &= \frac{1}{c}\int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} \exp\{-y^2\}\, dy\right\} \exp\{-x^2\}\, dx \\ &= \frac{1}{c}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-(x^2+y^2)\}\, dy\, dx \end{aligned}$$

Now, we can make a transformation to polar coordinates in this double integral, that is, set

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

for which the Jacobian or "change of variables" term is the determinant

$$\left|\begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}\right| = \left|\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right| = r\cos^2\theta + r\sin^2\theta = r$$

so that the integral above becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-(x^2+y^2)\}\,dy\,dx = \int_0^{\infty}\int_{-\pi}^{\pi} \exp\{-(r^2\cos^2\theta + r^2\sin^2\theta)\}\,r\,d\theta\,dr$$

$$= \int_0^{\infty}\int_{-\pi}^{\pi} \exp\{-r^2\}\,r\,d\theta\,dr$$

$$= \left\{\int_0^{\infty} \exp\{-r^2\}\,r\,dr\right\}\left\{\int_{-\pi}^{\pi} d\theta\right\}$$

$$= \left\{\left[-\frac{1}{2}\exp\{-r^2\}\right]_0^{\infty}\right\}\{2\pi\}$$

$$= \pi$$

and hence we deduce that

$$c = \frac{\pi}{c} \quad \text{so that} \quad c = \sqrt{\pi}$$

∴ Could define pdf f_X by

$$f_X(x) = \sqrt{\frac{1}{\pi}}\,\exp\{-x^2\}$$

or

$$f_X(x) = \sqrt{\frac{\lambda}{\pi}}\,\exp\{-\lambda x^2\}$$

For a "standard" case, choose

$$\lambda = \frac{1}{2}$$

$$f_X(x) = \sqrt{\frac{1}{2\pi}}\,\exp\left\{-\frac{1}{2}x^2\right\}$$

This is the pdf of the STANDARD NORMAL or STANDARD GAUSSIAN distribution

CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

– not available in closed form

Often write $\phi(x)$ for $f_X(x)$
$\Phi(x)$ for $F_X(x)$

Some features of ϕ and Φ:

$$\phi(-x) = \phi(x) \qquad x > 0$$

$$\Phi(-x) = 1 - \Phi(x) \qquad x > 0.$$

– follow from symmetry.

EXPECTATION

$$E_{f_X}[X] = 0$$

(see handout)

(– pdf is symmetric about 0
⟹ Expectation is 0)

GENERALIZATION

Suppose $X \sim \text{Normal}(0,1)$

i.e. $$f_X(x) = \sqrt{\frac{1}{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$$

let $Y = \mu + \sigma X \qquad \sigma > 0$

(LOCATION / SCALE TRANSFORMATION)

To derive the pdf of Y:

$$F_Y(y) = P[Y \leq y]$$
$$= P[\mu + \sigma X \leq y]$$
$$= P\left[X \leq \frac{y-\mu}{\sigma}\right]$$
$$= F_X\left(\frac{y-\mu}{\sigma}\right)$$
$$\Rightarrow f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right)$$
$$= \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{\frac{-1}{2\sigma^2}(y-\mu)^2\right\}$$

This is the pdf of the general Normal pdf.

$$Y \sim \text{Normal}(\mu, \sigma^2)$$

NOTE $F_Y(y) = \Phi\left(\frac{y-\mu}{\sigma}\right)$

$\therefore$ only need to compute standard Normal pdf

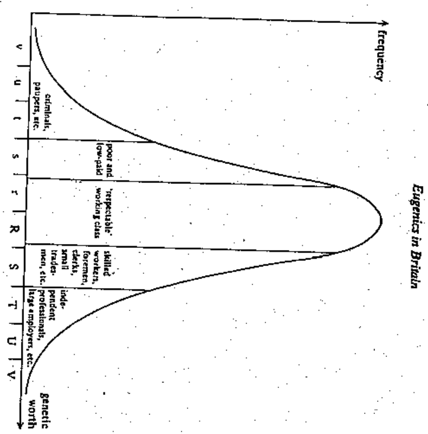
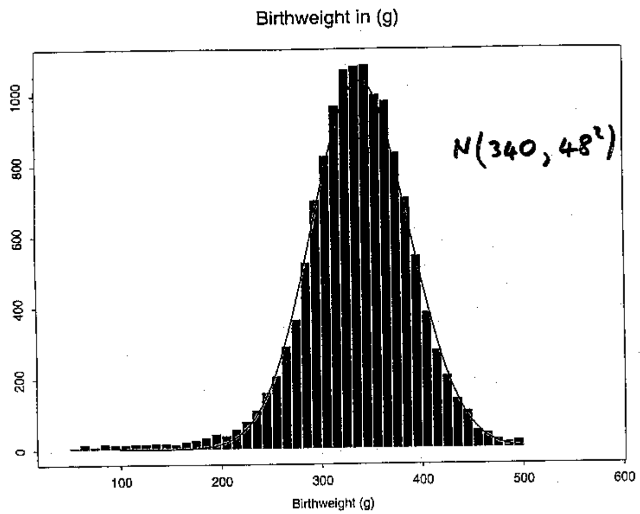


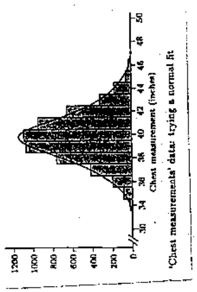
Figure 1. Galton's view of British social structure

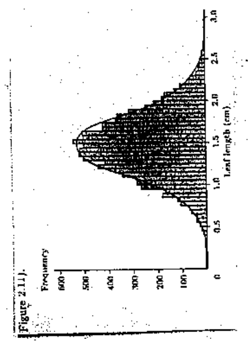
EXPECTATION

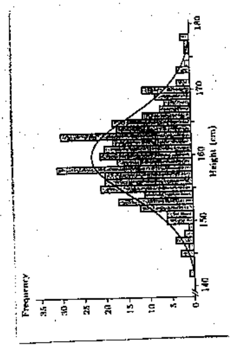
$$E_{f_Y}[Y] = \mu$$

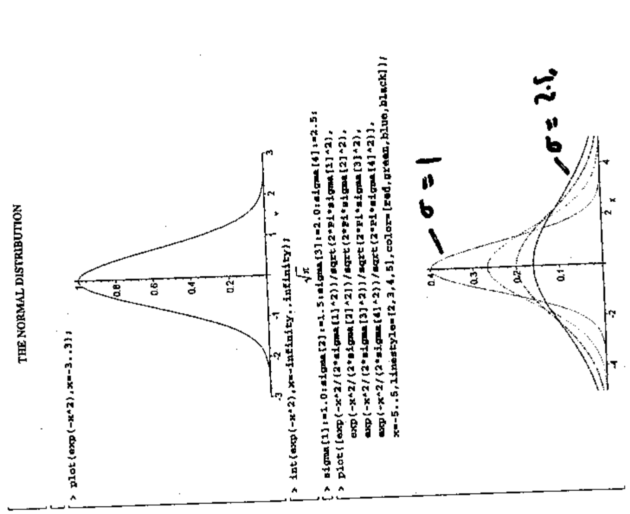
(general Normal pdf is symmetric about μ).











4.3 THE NORMAL DISTRIBUTION

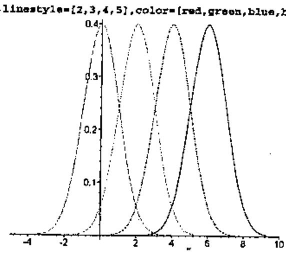
The limit proposed by Poisson was not the only, or even the first, approximation t the binomial: DeMoivre had already derived a quite different one in his 1718 trac *Doctrine of Chances.* Like Poisson's work, DeMoivre's theorem did not initiall attract the attention it deserved; it did catch the eye of Laplace, though, wh generalized it and included it in his influential *Théorie Analytique des Probabilité* published in 1812.

[redacted]

$$X \sim Bin(n,p): \lim_{n \to \infty} P\left(c < \frac{X-np}{\sqrt{npq}} < d\right) = \frac{1}{\sqrt{2\pi}} \int_c^d e^{-x^2/2}\,dx$$

```
> mu[1]:=0.0:mu[2]:=2.0:mu[3]:=4.0:mu[4]:=6.0:
> plot([exp(-(x-mu[1])^2/(2))/sqrt(2*Pi),
        exp(-(x-mu[2])^2/(2))/sqrt(2*Pi),
        exp(-(x-mu[3])^2/(2))/sqrt(2*Pi),
        exp(-(x-mu[4])^2/(2))/sqrt(2*Pi)],

  x=-5..10,linestyle=[2,3,4,5],color=[red,green,blue,black]);
```

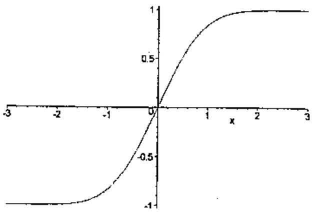


Description:

- The error function is defined for all complex x by

```
erf(x) = 2/sqrt(Pi) * int(exp(-t^2), t=0..x)
```

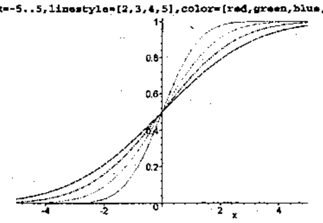
```
> plot(erf(x),x=-3..3);
```




```
> Phi[X] := x -> int(exp(-t^2/2)/sqrt(2*Pi),t=-infinity..x);
```

$$\Phi_X := x \to \int_{-\infty}^{x} \frac{e^{(-1/2\,t^2)}}{\sqrt{2\pi}}\,dt$$

```
> plot([Phi[X](x/sigma[1]),Phi[X](x/sigma[2]),
        Phi[X](x/sigma[3]),Phi[X](x/sigma[4])],
        x=-5..5,linestyle=[2,3,4,5],color=[red,green,blue,black]);
```




```
> seq(evalf(Phi[X](x/10)),x=-20..20);
```

.02275013194, .02871655989, .03593031918, .04456546274, .05479929170, .06680720100,
.08075665905, .09680048470, .1150696702, .1356660610, .1586552539, .1840601253,
.2118553986, .2419636521, .2742531177, .3085375386, .3445782582, .3820885776,
.4207402902, .4601721624, .5000000000, .5398278370, .5792597090, .6179114220,
.6554217410, .6914624610, .7257468815, .7580363475, .7881446010, .8159398745,
.8413447455, .8643339385, .8849303290, .9031995145, .9192433405, .9331927985,
.9452007075, .9554345365, .9640696800, .9712834395, .9772498675

```
>
```