CHAPTER 2

2. PROBABILITY : DEFINITIONS, INTERPRETATIONS, LAWS.

Objective: to attach a numerical value to event, in order to quantify the chance that it occurs.

2.1 THE MEANING OF PROBABILITY

Mathematical Definition:

For sample space Ω, let

Σ - set of subsets of Ω.

Define the PROBABILITY FUNCTION $P(\,.\,)$ by

$$\begin{aligned} P : \Sigma &\longrightarrow \mathbb{R} \\ E &\longmapsto x \end{aligned}$$

i.e. For event $E \subseteq \Omega$

$$P(E) = x$$

for some $x \in \mathbb{R}$

i.e $P(\,.\,)$ is a function (map) from Σ to $\mathbb{R}$

i.e P maps event E to real number x.

so P is a function in the same way that

$$f(x) = e^x$$
$$f(x) = \cos x \quad \ldots$$

are functions EXCEPT that P acts on sets (events)

i.e

P is a SET-FUNCTION

This definition does <u>not</u> specify <u>how</u> (numerically) probabilities are assigned.

There are three common interpretations

(1) <u>RELATIVE FREQUENCY</u>

Consider n_{TOT} identical replications of the experiment.....

(n_{TOT} large)

Let n_E be the number of times that event E occurs (out of the total n_{TOT})

Then

$$P(E) = \lim_{n_{TOT} \to \infty} \frac{n_E}{n_{TOT}}.$$

<u>EXAMPLE</u> Single toss of drawing pin

Sample outcomes : $\perp$,

$\{0 \; , \; 1\}$

Event E : "outcome is $\perp$"

What is $P(E)$ here ?

- could use "relative frequency" definition.

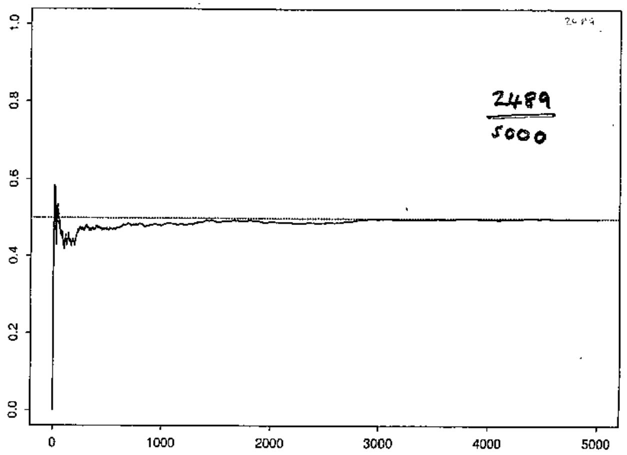
But

(a) the "thought experiment" $n_{TOT} \longrightarrow \infty$ is difficult to conceive.

(b) any practical version (n_{TOT} finite) may not be representative

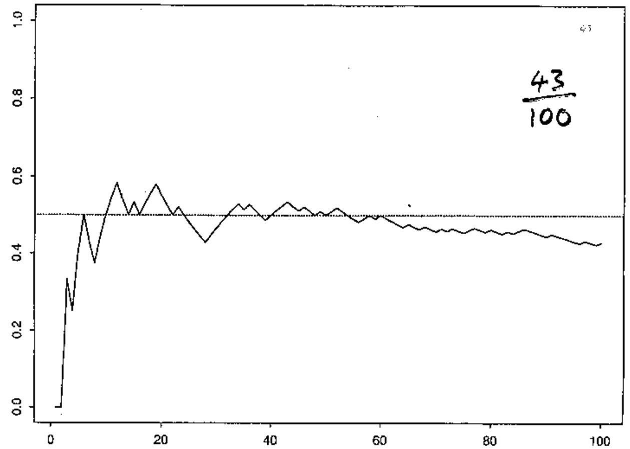
$\longrightarrow$ becomes a statistical issue.

5000 Coin tosses: relative frequency of Heads



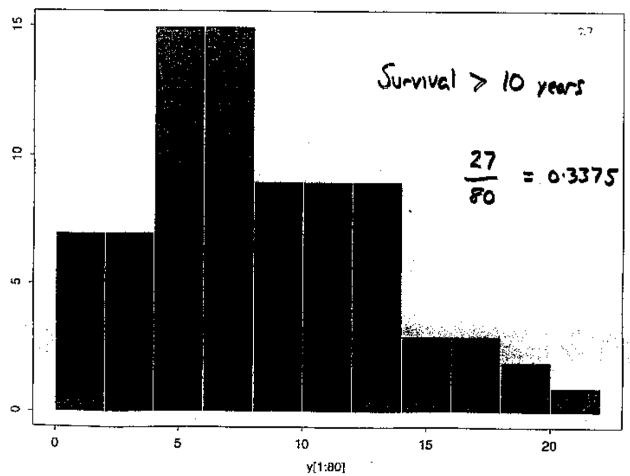
0.4978

100 Coin tosses: relative frequency of Heads

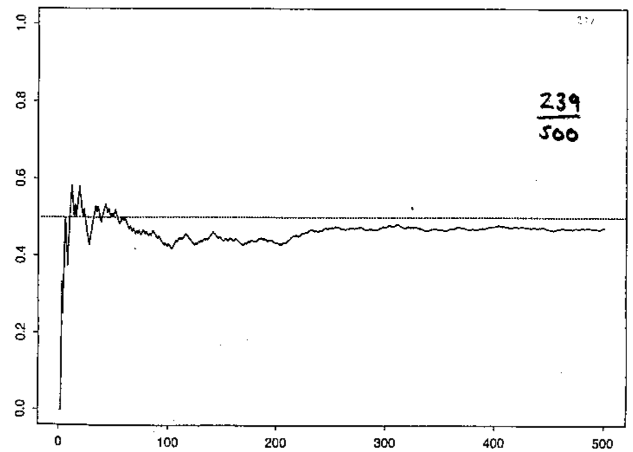


0.43

Survival Times of 80 patients



500 Coin tosses: relative frequency of Heads



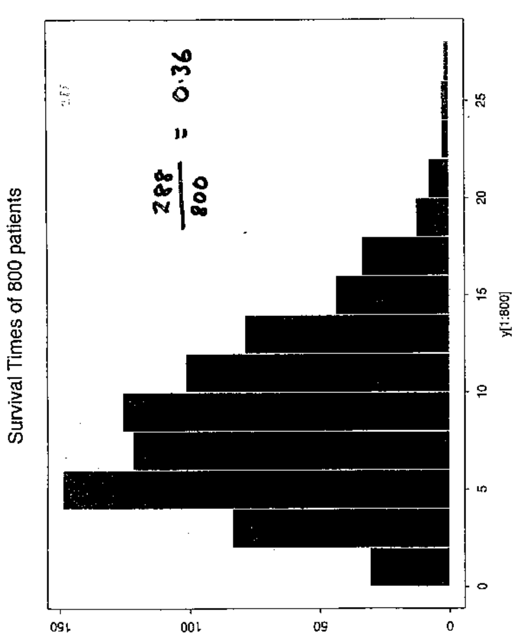
0.478

(II) CLASSICAL

In the special case where all of the sample outcomes are equally likely :

If there are n_Ω sample outcomes in total, and n_E outcomes in the event E, then define

$$P(E) = \frac{n_E}{n_\Omega}$$



Two points :

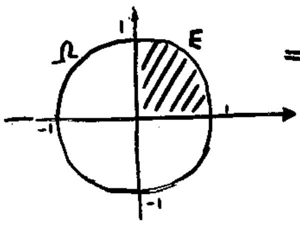
(I) Ω must be finite

If Ω is UNCOUNTABLE :

e.g. E is some interval of $\mathbb{R}$ or region of $\mathbb{R}^k$

should define

$$P(E) = \frac{\text{Area of } E}{\text{Area of } \Omega} = \frac{A_E}{A_\Omega}.$$



$\Rightarrow P(E) = \frac{1}{4}$

(II) May be difficult to enumerate n_E or n_Ω

(or evaluate A_E, A_Ω)

e.g. Lottery draws :

- 6 balls from 49 without replacement
- all subsets of 6 balls equally likely

How many possible draws? n_Ω?
How many draws would win you £10 for a given ticket?

Otherwise, the classical definition is useful for simple mechanical experiments

COINS
DICE
CARDS
LOTTERY

– not so much use for general "real-world" situations.

Quantitatively: assign

$$P(E) = x$$

if you regard xM to be a "fair" price to pay to enter the following game:

E occurs $\boxed{\text{WIN } M}$!

E does not occur $\boxed{\text{WIN } 0}$ .....

"FAIR" – not guaranteed to win or to lose money.

(iii) SUBJECTIVE

For event E, assign $P(E)$ according to your own personal "degree of belief" in E, given experimental circumstances

EXAMPLE

EXPERIMENT: Noon time temp. tomorrow measurement

$$\Omega : \{ x : -20 \leq x \leq 25 \}$$

$$E : \{ x : 0 \leq x \leq 5 \}$$

Could assign $P(E)$ according to

- historical information
- local knowledge

but will not and need not be the same for all individuals.

May be difficult to implement in practice, but is a valid (and universal) interpretation of probability.

Each of these interpretations depend on assumptions about experimental conditions; denote these assumptions by $\mathcal{E}$

Now need to study how the function $P(\ .\ )$ behaves

MATHEMATICALLY

2.2 MATHEMATICAL RULES OF PROBABILIT

What properties must the set function $P(.)$ exhibit in order to give a consistent representation of ideas about "UNCERTAINTY"?

THE PROBABILITY AXIOMS

For events $E, F \subseteq \Omega$

(I) $0 \leq P(E) \leq 1$

(II) $P(\Omega) = 1$

(III) If $E \cap F \equiv \phi$, then $P(E \cup F) = P(E) + P(F)$

EXTENSIONS TO AXIOM III

If $E_1, .., E_k$ are (pairwise) mutually exclusive $E_i \cap E_j \equiv \phi \quad \forall i, j$

$$P\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} P(E_i)$$

k infinite:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

(III) - "THE ADDITION RULE"

EXTENSIONS: "COUNTABLE ADDITIVITY"

(The axioms can be deduced as a minimal set of rules for $P(\ .\ )$ in each of the three interpretations)

2.3 COROLLARIES TO THE AXIOMS

(i) $P(E') = 1 - P(E)$

$\Omega \equiv E \cup E' \quad (E \cap E' \equiv \phi)$

$\Rightarrow P(\Omega) = P(E \cup E')$

LHS: $P(\Omega) = 1$ AX (II)

RHS: $P(E \cup E') = P(E) + P(E')$ AX (III

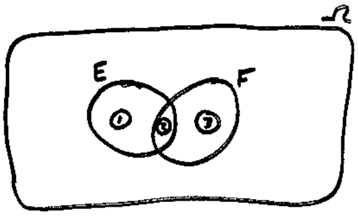
Hence result.

(iii) For any $E, F \subseteq \Omega$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Proof Ex 2 Qu 4b.

Hint : visual aid



$$E \cup F \equiv \underset{①}{(E \cap F')} \cup \underset{②}{(E \cap F)} \cup \underset{③}{(E' \cap F)}$$

or $E \cup F \equiv E \cup (E' \cap F)$

Extensions :

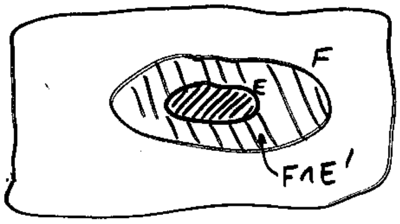
- $P(E \cup F \cup G) = P(E) + P(F) + P(G)$
  $- P(E \cap F) - P(E \cap G) - P(F \cap G)$
  $+ P(E \cap F \cap G)$

- general formula for
  $P(E_1 \cup E_2 \cup \ldots \cup E_k)$
  – Ex 2 : Q7

" The General Addition Rule "

(ii) $E \subseteq F \Rightarrow P(E) \leq P(F)$

$$F \equiv E \cup (F \cap E')$$



$\Rightarrow P(F) = P(E) + P(F \cap E')$ III
$\geqslant P(E)$ I

Could display probabilities in a table

	E	E'	TOTAL
F	$P_{E \cap F}$	$P_{E' \cap F}$	P_F
F'	$P_{E \cap F'}$	$P_{E' \cap F'}$	$P_{F'}$
TOTAL	P_E	$P_{E'}$	P_Ω

i.e $P(E) = P(E \cap F) + P(E \cap F')$

$P_E = P_{E \cap F} + P_{E \cap F'}$

$E \equiv (E \cap F) \cup (E \cap F')$ etc.

– table defines a four-way partition of Ω.

If Ω is FINITE, with all outcomes equally likely, the following table is an equivalent representation:

	E	E'	TOTAL
F	$n_{E\cap F}$	$n_{E'\cap F}$	n_F
F'	$n_{E\cap F'}$	$n_{E'\cap F'}$	$n_{F'}$
TOTAL	n_E	n_E	n_Ω

i.e. there are $n_{E\cap F}$ outcomes in $E \cap F$ etc.

NUMERICAL EXAMPLES

EXAMPLE

Suppose $P(E) = \frac{1}{3}$

$P(F) = \frac{1}{2}$

$P(E \cup F) = \frac{3}{4}$

	E	E'	TOT.
F	p_{11}	p_{12}	$\frac{1}{2}$
F'	p_{21}	p_{22}	$\frac{1}{2}$
TOT.	$\frac{1}{3}$	$\frac{2}{3}$	

We have

$$p_{11} + p_{21} = \frac{1}{3}$$
$$p_{12} + p_{22} = \frac{2}{3}$$

COLUMNS

$$p_{11} + p_{12} = \frac{1}{2}$$
$$p_{21} + p_{22} = \frac{1}{2}$$

ROWS

Solve simultaneously for $p_{11}, p_{12}, p_{21}, p_{22}$.

GENERAL ADDITION RULE:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$\Leftrightarrow P(E\cap F) = P(E) + P(F) - P(E \cup F)$$
$$= \frac{1}{3} + \frac{1}{2} - \frac{3}{4}$$
$$= \frac{1}{12}$$

$P(E' \cap F')$: Recall De Morgan

$$E' \cap F' \equiv (E \cup F)'$$

$$\Rightarrow P(E'\cap F') = P((E\cup F)') = 1 - P(E \cup F) = \frac{1}{4}$$

Similarly, for $P(E' \cup F')$:

$$E' \cup F' \equiv (E \cap F)'$$

$$\Rightarrow P(E'\cup F') = P((E\cap F)') = 1 - P(E\cap F)$$
$$= \frac{11}{12}$$

Finally for $P(E' \cap F)$:

"Row 1" $\Rightarrow F \equiv (E\cap F) \cup (E' \cap F)$

$$\Rightarrow P(F) = P(E\cap F) + P(E'\cap F)$$
$$\Rightarrow P(E'\cap F) = P(F) - P(E \cap F)$$
$$= 5/12$$

i.e.

	E	E'	TOT
F	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{2}$
F'	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{1}{2}$
TOT	$\frac{1}{3}$	$\frac{2}{3}$	1

EXAMPLE 2

Suppose $P(E) = p$

$P(E') = p^2$

What is p?

Require $P(E) + P(E') = 1$

$\Rightarrow \quad p + p^2 = 1$

$\therefore$ must solve $p^2 + p - 1 = 0$ for $0 \le p \le 1$

EXAMPLE 3

In a sports club:

36	play	Tennis	T
28	play	Squash	S
18	play	Badminton	B
22	play	Tennis + Squash	$T \cap S$
12	play	Tennis + Badminton	$T \cap B$
9	play	Squash + Badminton	$S \cap B$
4	play	All Three sports	$T \cap S \cap B$

How many people play atleast one sport?

– Use "general addition rule".....