CHAPTER 1

1. SAMPLE SPACES AND EVENTS

Need to introduce

TERMINOLOGY NOTATION

via which to handle UNCERTAINTY

UNCERTAINTY - the absence of perfect knowledge of the "state of nature"

EXAMPLES

1. (i) "What is the 1,000,000th digit of π?" 3.1415926....

   (ii) "What is the height of this building?"

   (iii) "How many people are in this room?"

2. "Will the coin land as HEADS?"

We will treat uncertainty in 1 and 2 in identical fashion.

"State of nature"?

- some aspect of the real world
- could be the current state 1

or

could be the result of a procedure or experiment that is yet to be carried out. 2

1. IMPERFECT OBSERVATION
2. OBSERVATION YET TO BE CARRIED OUT.

1.1 REPRESENTING UNCERTAINTY

First step in assessing "probable" outcomes / states of nature :

LIST THE POSSIBLE OUTCOMES

- depends on experimental context.
- crucially affects subsequent assessment

In most cases, the list of outcomes will be essentially equivalent to some set of real numbers

EXAMPLES

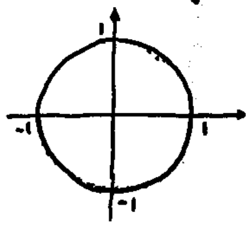
COUNTING EXPERIMENT	0 1 2 3 ...
MEASUREMENT	Min — Max
COINS	$\{H, T\} \equiv \{0, 1\}$
Toss a coin	$\{H, T\}$
Toss a coin until a H is obtained	$\{H, TH, TTH, ...\}$

Usually we consider individual outcomes, but sometimes consider collections of outcomes

e.g. "HISTOGRAM"

- considers the outcomes that are "binned" together.

EXPERIMENT	POSSIBLE OUTCOMES
Roll of a single die	$\{1,2,3,4,5,6$
Roll of two dice	$\{(1,1),(1,2)....$ $...,(6,6)\}$
Arrow into target	$\{(x,y): x^2+y^2<1$



1.2 MANIPULATING COLLECTIONS OF SAMPLE OUTCOMES

The collection of possible outcomes of an experiment forms a SET

NOTATION

$$A \equiv \{a_1, a_2, \ldots, a_k\}$$

$a \in A$ a is a member of A

$a \notin A$ a is not a member.

The sample space Ω could be a

FINITE LIST e.g ROLL OF DIE

INFINITE LIST TOSSES UNTIL FIRST HEAD

CONTINUUM OF POINTS – REGION OF $\mathbb{R}^k$ HEIGHT/WEIGHT MEASUREMEN

"LISTS" – COUNTABLE SET

"REGIONS OF $\mathbb{R}^k$" – UNCOUNTABLE SET

SAMPLE SPACE

- the set of all possible outcomes
- denoted Ω

SAMPLE/ELEMENTARY OUTCOME

- a single possibility
- denoted ω

$$\Omega \equiv \{\omega_1, \omega_2, \ldots\}$$

$\omega_1, \omega_2, \ldots$ are elements of Ω.

Subsets of the set Ω are called

EVENTS

i.e. an event is a collection of sample outcomes

e.g. $E \equiv \{\omega_3, \omega_7, \ldots\}$

e.g. ROLL OF DIE "ODD" $\equiv \{1,3,5\}$

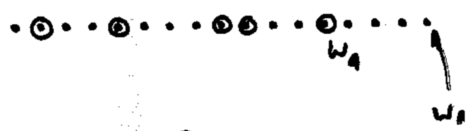
"HISTOGRAM" "Bin i" $\equiv \{x \in B_i\}$

If

ω_A – actual outcome

Then event E occurs if $\omega_A \in E$

E does not occur if $\omega_A \notin E$



$$E = \{ \odot \}$$

SPECIAL CASES OF EVENTS

For all experiments

Ω - the CERTAIN EVENT

$\omega_A \in \Omega$

$\omega_A \notin \phi$

ϕ - the IMPOSSIBLE EVENT

the empty set

the set containing zero sample outcomes

Further notation : for two events E, F

$E \subseteq F$: $\omega \in E \Rightarrow \omega \in F$

$E \subset F$: $E \subseteq F$, but $E \not\equiv F$

"E is a subset of F"

"E is contained in F"

1.3 OPERATIONS OF SET THEORY

Events in probability theory are manipulated using set theory operations.

For events E, F we use

COMPLEMENT E'

UNION $E \cup F$

INTERSECTION $E \cap F$

COMPLEMENT $E' \equiv \Omega \setminus E$

$E' \equiv \{\omega \in \Omega : \omega \notin E\}$

- the set of sample outcomes that are in Ω but not in E.

e.g. $\Omega' \equiv \phi$

so

E' occurs $\Leftrightarrow$ E does not occur

INTERSECTION

$$E \cap F \equiv \{\omega \in \Omega : \omega \in E \ \underline{\text{and}}\ \omega \in F\}$$

– the collection of sample outcomes that are in E <u>and</u> in F

E	•	•	⊙	•	⊙	⊙	•	•
F	•	⊙	•	•	⊙	⊙	•	⊙
$E \cap F$	•	•	•	•	⊙	⊙	•	•

$E \cap F$ occurs

$\iff$ E occurs and F occurs

SPECIAL CASE

$$E \cap F \equiv \emptyset$$

– say that E and F are **MUTUALLY EXCLUSIVE** events.

UNION

$$E \cup F \equiv \{\omega \in \Omega : \omega \in E \text{ or } \omega \in F \text{ or } \omega \in E \cap F\}.$$

– The set of sample outcomes that are in E, or in F, or in both.

E	•	•	⊙	•	⊙	⊙	•	•
F	•	⊙	•	•	⊙	⊙	•	⊙
$E \cup F$	•	⊙	⊙	•	⊙	⊙	•	⊙

$E \cup F$ occurs

$\iff$ E occurs, or F occurs, or <u>both</u> occur.

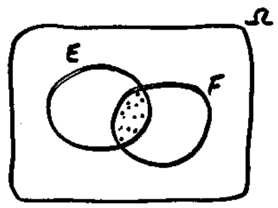
SPECIAL CASE

$$E \cup F \equiv \Omega$$

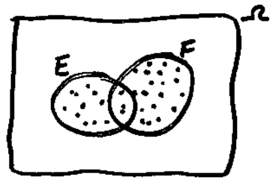
– say that E and F are **EXHAUSTIVE** events.

Useful visual representation

VENN DIAGRAM



$E \cap F$



$E \cup F$

$$E \equiv F \iff \omega \in E \iff \omega \in F$$

Some elementary results:

(i) $E \cap F \subseteq E$

$E \cap F \subseteq F$

(ii) $E \subseteq E \cup F$

$F \subseteq E \cup F$

(iii) DE MORGAN'S LAWS

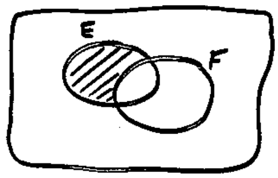
(a) $(E \cup F)' \equiv E' \cap F'$

(b) $(E \cap F)' \equiv E' \cup F'$

Note: Please only use

$'$, $\cap$, $\cup$

and not "$+$" or "$-$"



Tempting to write $E - (E \cap F)$

– not valid!

Should write $E \cap F'$

NOTE

UNION / INTERSECTION are

"binary operations"

(take two arguments)

– need to extend to three or more

$E, F, G, \ldots$

$E \cup (F \cup G)$

We know that, for real numbers a, b, c we have the rules

$$a(b+c) = ab + ac$$

$$(a+b)(c+d) = ac + ad + bc + bd$$

- the "DISTRIBUTIVE LAWS" tell us how to combine $\cup$ and $\cap$ in a similar manner.

EXAMPLE Proof of De Morgan law

$$(E \cup F)' \equiv E' \cap F'$$

Let $A \equiv E \cup F$

$B \equiv E' \cap F'$

Must show $A' \equiv B$

i.e. (i) $A \cap B \equiv \emptyset$

(ii) $A \cup B \equiv \Omega$

(i) $A \cap B \equiv (E \cup F) \cap B$

$\equiv (E \cap B) \cup (F \cap B)$

But $E \cap B \equiv E \cap (E' \cap F') \equiv \emptyset$

$F \cap B \equiv F \cap (E' \cap F') \equiv \emptyset$

$\therefore A \cap B \equiv \emptyset \cup \emptyset \equiv \emptyset$ $\square$

ii) $A \cup B \equiv A \cup (E' \cap F')$

$= (A \cup E') \cap (A \cup F')$

But $A \cup E' \equiv (E \cup F) \cup E' \equiv \Omega$

$A \cup F' \equiv (E \cup F) \cup F' \equiv \Omega$

$\therefore A \cup B \equiv \Omega \cap \Omega \equiv \Omega$ $\square$

Now can express / combine statements involving "possible" outcomes

The next step is to find simple representations of complex events

and if possible to find the simplest representation relevant to probability specifications / calculations.

QUIET PLEASE!

OBJECTIVE: Express events in complex experiments via simpler events

EXAMPLE

Machine M

Three components C_1, C_2, C_3

M only functions if at least two of C_1, C_2, C_3 function.

let $W \equiv$ "machine works"

$C_i \equiv$ "component i works"

Then

$$W \equiv (C_1 \cap C_2) \cup (C_1 \cap C_3) \cup (C_2 \cap C_3) \cup (C_1 \cap C_2 \cap C_3)$$

Is this the best representation?

Is it better to express as

$$W \equiv (C_1 \cap C_2 \cap C_3') \cup (C_1 \cap C_2' \cap C_3) \cup (C_1' \cap C_2 \cap C_3) \cup (C_1 \cap C_2 \cap C_3) \quad ?$$

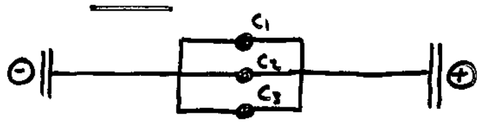
- These events are mutually exclusive

[We will see that this is a more useful expression for probability calculations]

Can think of components arranged in a "network" or "circuit"

e.g. "SERIES" CONNECTION

"PARALLEL" CONNECTION



MODEL : Circuit functions if there is a fully functioning path from ⊖ to ⊕

i.e. for SERIES

$$W \equiv C_1 \cap C_2 \cap C_3$$

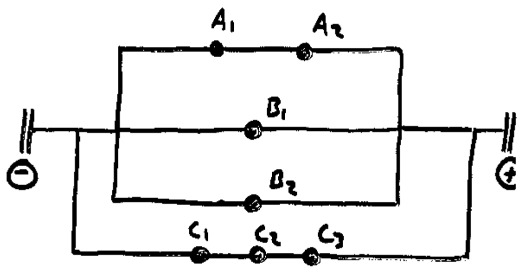
"ALL WORK"

for PARALLEL

$$W \equiv C_1 \cup C_2 \cup C_3$$

"AT LEAST ONE WORKS"

For complex circuits :



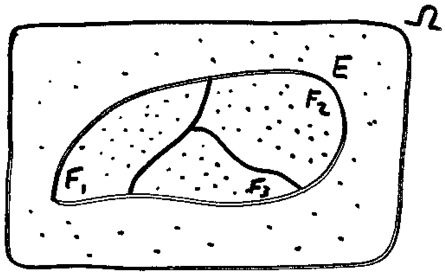
$$W \equiv S_1 \cup S_2$$

where $S_1 \equiv (A_1 \cap A_2) \cup B_1 \cup B_2$

$S_2 \equiv C_1 \cap C_2 \cap C_3$

PARTITIONING

Visual representation



Event E broken down into three subsets F_1, F_2, F_3

DEFINITION

Suppose events $E, F_1, F_2, \ldots, F_k$ in Ω exist such that

(a) $F_i \cap F_j \equiv \phi \quad \forall i, j$

(b) $\bigcup_{i=1}^{k} F_i \equiv E$.

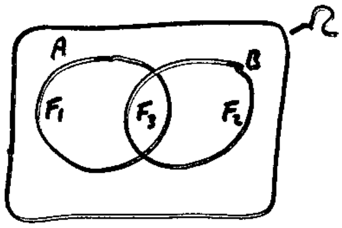
(a) : $F_1, \ldots, F_k$ are PAIRWISE MUTUALLY EXCLUSIVE

(b) : $F_1, \ldots, F_k$ are EXHAUSTIVE for E

Then $F_1, F_2, \ldots, F_k$ form a **PARTITION** of E

EXAMPLE For general events A, B

$$A \cup B \equiv (A \cap B') \cup (A' \cap B) \cup (A \cap B)$$
$$\equiv F_1 \cup F_2 \cup F_3$$



EXAMPLE Histogram : Bin i

$$F_i \equiv [i-1, i) \subseteq \mathbb{R}^+$$

$i = 1, 2, \ldots$

PARTITIONS are crucial in simplification of probability calculations

- we will see that if $F_1, F_2, \ldots F_k$ form a partition of E, then the probability of E can be evaluated by summation of the probabilities of $F_1, F_2, \ldots F_k$.