M1S : EXERCISES 9

TRANSFORMATIONS OF RANDOM VARIABLES

1. Suppose that X is a continuous random variable with range $\mathbb{X} = [0,1]$, and probability density function f_X specified by

$$f_X(x) = 2(1-x) \qquad 0 \leq x \leq 1$$

and zero otherwise. Find the probability distributions of random variables Y_1, Y_2 and Y_3 defined respectively by

$$\text{(i)} \quad Y_1 = 2X - 1 \qquad \text{(ii)} \quad Y_2 = 1 - 2X \qquad \text{(iii)} \quad Y_3 = X^2$$

that is, in each case, find the range and the density function.

2. The annual profit (in millions of pounds) of a manufacturing company is a function of product demand. If X is the continuous random variable corresponding to the demand in a given year, then the annual profit is also a continuous random variable, Y say, where

$$Y = 2(1 - e^{-2X})$$

If X has an Exponential distribution with parameter $\lambda = 6$, find the expected annual profit.

3. The continuous random variable X has a Uniform distribution on the interval [-1,1]. Find the probability density function of random variables

$$\text{(a)} \quad Y = |X| \qquad \text{(b)} \quad Z = X^2$$

4. If X is **any** continuous random variable with distribution function F_X, show that

(i) Random variable $U = F_X(X)$ has a Uniform distribution on $[0,1]$;
(ii) Random variable $Y = -\log F_X(X)$ has an exponential distribution

5. If X is a continuous random variable on range $\mathbb{X} \equiv \mathbb{R}^+$ with probability density function specified by

$$f_X(x) = \alpha\beta x^{\alpha-1} e^{-\beta x^\alpha} \qquad x > 0$$

and zero otherwise, for parameters $\alpha, \beta > 0$, then X has a *Weibull* distribution. Show that $Y = X^\alpha$ has an exponential distribution.

6. If X has a Geometric distribution with parameter θ, show using the moment generating function of X or otherwise that

$$\mathrm{E}_{f_X}[X] = \frac{1}{\theta} \qquad \mathrm{Var}_{f_X}[X] = \frac{1-\theta}{\theta^2}$$

Hence deduce the forms of the expectation and variance of a negative binomial distribution with parameters n and θ.

7. Let X be a discrete random variable with range $\mathbb{X} = \{0, 1, 2, ...\}$. Show that

$$\mathrm{E}_{f_X}[X] = \sum_{x=0}^{\infty} \mathrm{P}[\ X > x\ ]$$

8. Let X be a continuous random variable with range $\mathbb{X}$ and probability density function f_X. Let g be a real-valued function whose range includes $\mathbb{X}$, and let random variable Y be defined by $Y = g(X)$. Prove that

$$\mathrm{E}_{f_Y}[Y] = \mathrm{E}_{f_X}[g(X)]$$

provided that both expectations exist.

9. Suppose that random variable X has a standard normal distribution.

(i) Find the cumulative distribution function (cdf) of $Y = X^2$ in terms of the standard normal cdf Φ.

Hint: for the cdf of Y, we have

$$\mathrm{P}[\, Y \leq y \,] \quad \equiv \mathrm{P}\left[\, X^2 \leq y \,\right] \equiv \mathrm{P}[\, |X| \leq \sqrt{y} \,]$$

(ii) Find the probability density function of Y, f_Y.

(iii) Identify (by name) the probability distribution of Y.

10. Suppose now that X_1 and X_2 are independent and identically distributed random variables, each having a standard normal distribution. Let random variable V be defined by

$$V = X_1^2 + X_2^2$$

Find the pdf of V.