M1S : EXERCISES 8
CONTINUOUS PROBABILITY DISTRIBUTIONS : PROPERTIES AND FUNCTIONALS

1. Probability models for random variables taking values on $\mathbb{R}^+$ may be specified via a number of probability functions: for real valued x

Continuous CDF	$F_X(x) = \mathrm{P}[X \leq x]$
PDF	$f_X(x) = \dfrac{d}{dt}\{F_X(t)\}_{t=x}$
RELIABILITY or SURVIVOR function	$R_X(x) = \mathrm{P}[X > x]$
HAZARD function	$h_X(x) = \dfrac{f_X(x)}{R_X(x)}$
CUMULATIVE or INTEGRATED HAZARD function	$H_X(x) = \displaystyle\int_0^x h_X(t)\,dt$

Verify that

(i) $R_X(x) = 1 - F_X(x)$ (ii) $R_X(x) = \exp\{-H_X(x)\}$ (iii) $f_X(x) = h_X(x)\exp\{-H_X(x)\}$

Typically, these different formulations are utilized in the context of failure data, where, for example, the time until failure of some mechanism is to be measured. In this context, we may interpret of the pdf at x as quantifying the instantaneous *rate* at which failures occur at a given x. In light of this interpretation, explain, *in conditional probability terms*, the interpretation of the hazard function.

2. Using the definitions in 1, find the cumulative distribution, reliability, hazard and integrated hazard functions for the following pdfs defined on $\mathbb{X} = \mathbb{R}^+$ for parameters $\alpha > 0, \beta > 0$:

(i) $f_X(x) = \alpha x^{\alpha-1}\exp\{-x^\alpha\}$ (ii) $f_X(x) = \dfrac{\alpha x^{\alpha-1}}{(1+x^\alpha)^2}$ (iii) $f_X(x) = \dfrac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}$

3. *EXPECTATION*

For a continuous random variable X with range $\mathbb{X}$, the **expectation** is defined as

$$E_{f_X}[X] = \int_{\mathbb{X}} x f_X(x)\,dx$$

analogously to the definition in the discrete case, again interpreted as a "centre of probability density", or as weighted average of values in $\mathbb{X}$ with weights determined by the pdf. Note that if the integral is not convergent (or, less formally, infinite), the expectation is "not defined" or "does not exist". Calculate the expectations for the following pdfs with ranges given:

(i)	$f_X(x)$	$= \lambda e^{-\lambda x}$	$x > 0$	$\lambda > 0$
(ii)	$f_X(x)$	$= \dfrac{\lambda}{2} e^{-\lambda\lvert x\rvert}$	$x \in \mathbb{R}$	$\lambda > 0$
(iii)	$f_X(x)$	$= \dfrac{\lambda^{k+1}}{k!} x^k e^{-\lambda x}$	$x > 0$	$\lambda > 0, k = 1,2,3,...$
(iv)	$f_X(x)$	$= \dfrac{\alpha\beta^\alpha}{(\beta+x)^{\alpha+1}}$	$x > 0$	$\alpha, \beta > 0$
(v)	$f_X(x)$	$= \dfrac{1}{\pi}\dfrac{1}{1+x^2}$	$x \in \mathbb{R}$	
(vi)	$f_X(x)$	$= \dfrac{(m+n+1)!}{m!n!} x^m (1-x)^n$	$0 \leq x \leq 1$	$m, n = 1,2,3,..$

4. *A GENERATING FUNCTION FOR CONTINUOUS VARIABLES*

If X is a continuous rv with pdf f_X, consider the function $G_X(t)$ defined by

$$G_X(t) = \int_{\mathbb{X}} t^x f_X(x)\, dx$$

(that is, as in the probability generating function for a discrete variable, with the sum replaced by an integral). This function is **not** a generating function for probabilities, but it does have useful properties.

Prove that, in this continuous case,

$$\frac{d}{dt}\left\{G_X(t)\right\}_{t=1} = E_{f_X}[X]$$

Prove further that

$$\frac{d^r}{dt^r}\left\{G_X(t)\right\}_{t=1} = \int_{\mathbb{X}} x(x-1)(x-2)...(x-r+1)f_X(x)\, dx$$

(*you may assume that the integral defining $G_X(t)$ is convergent, and hence it is valid to differentiate under the integral sign*)

5. *MOMENTS*

The concept of expectation as measure of "centre of probability mass" or "centre of probability density" can be generalised to other include other measures. The first extension is that to *moments*; we define the *rth moment* of a distribution, denoted $\mathrm{E}_{f_X}[X^r]$, for $r = 1, 2, 3, ...$ by

$$E_{f_X}[X^r] = \begin{cases} \displaystyle\sum_{x \in \mathbb{X}} x^r f_X(x) & \text{if } X \text{ is } \textbf{discrete} \\ \displaystyle\int_{\mathbb{X}} x^r f_X(x)\, dx & \text{if } X \text{ is } \textbf{continuous} \end{cases}$$

The moments provide a means of description of a distribution; they are fixed constant values determined by the form of $f_X(x)$ that include the expectation as a special case ($r = 1$). Often, we write $\mathrm{E}_{f_X}[X^r] = m_r$.

Now, consider the function, $M_X(t)$, which is a functional of f_X defined by

$$M_X(t) = \int_{\mathbb{X}} e^{tx} f_X(x)\, dx$$

Prove, by differentiating under the integral sign, that

$$\frac{d^r}{dt^r}\left\{M_X(t)\right\}_{t=0} = m_r$$

Prove further, using a Taylor expansion of $M_X(t)$ about 0 that

$$M_X(t) = 1 + \sum_{r=1}^{\infty} \frac{m_r}{r!} t^r$$

and hence deduce that $M_X(t)$ is a generating function for the sequence $\{m_r/r!\}$ for $r = 1, 2, 3, ...$.

6. Compute the functions $G_X(t)$ (from Q4) and $M_X(t)$ (from Q5) for the pdfs specified for parameter $\lambda > 0$by

(i) $f_X(x) = \lambda e^{-\lambda x} \qquad x > 0$

(ii) $f_X(x) = \dfrac{\lambda^{k+1}}{k!} x^k e^{-\lambda x} \qquad x > 0$

(iii) $f_X(x) = \left(\dfrac{\lambda}{2\pi}\right)^{1/2} \exp\left\{-\lambda x^2/2\right\} \qquad x \in \mathbb{R}$

What is the functional relationship between $G_X(t)$ and $M_X(t)$?