M1S : EXERCISES 6

PROPERTIES OF DISCRETE PROBABILITY DISTRIBUTIONS

1. Show that the function

$$\frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^x$$

for parameter $\lambda > 0$ is a valid probability mass function for a discrete random variable X taking values on $\{0, 1, 2, ...\}$, and find the corresponding cumulative distribution function.

2. Suppose that discrete random variable X has a *negative binomial* distribution, that is, the probability mass function of X is given by

$$f_X(x) = \binom{x-1}{n-1}\theta^n(1-\theta)^{x-n} \qquad x = n, n+1, ...$$

and zero otherwise, for $n \geq 1$ and $0 < \theta \leq 1$. Recall that X corresponds to the number of binary trials required to obtain n successes.

Find the probability mass function of discrete random variable Y defined by $Y = X - n$.

3. It can be shown that, if X has a *Hypergeometric distribution*, $X \sim \textit{Hypergeometric}(N, R, n)$ (so that X is the count of the number of Type I items in a sample of size n), then if N and R are large, so that in particular $N, R >> n$, then

$$f_X(x) \approx \binom{n}{x}\theta^x(1-\theta)^{n-x} \qquad x = 0, 1, ..., n$$

and zero otherwise, where $\theta = R/N$, then X has an approximate Binomial distribution, with parameters n and θ. Explain this result by considering sampling without replacement from a finite but large population.

4. A fair coin is tossed n times. Let X be the discrete random variable corresponding to the difference between the number of heads and the number of tails observed. Find the range and probability mass function of X.

5. If X has a Geometric distribution with parameter θ, so that

$$f_X(x) = (1-\theta)^{x-1}\theta \qquad x = 1, 2, 3, ...$$

and zero otherwise, show that, for $n, k \geq 1$,

$$\mathrm{P}[\ X = n + k \mid X > n\ ] = \mathrm{P}[\ X = k\ ]$$

This result is known as the *Lack of Memory* property.

6. For which values of k and α are the following functions valid probability mass functions on the ranges given;

$$\text{(i) } f(x) = \frac{k}{x(x+1)}, \quad x = 1, 2, 3, ... \qquad \text{(ii) } f(x) = kx^\alpha, \quad x = 1, 2, 3, ...$$

7. Consider a sequence of *Bernoulli* random variables $X_1, ..., X_n$ each with parameter θ resulting from independent binary trials, so that

$$\mathrm{P}[X = 0] = 1 - \theta \qquad \mathrm{P}[X = 1] = \theta$$

Find the probability distributions of the random variables

$$\text{(i) } Y = \mathrm{Min}\{X_1, ..., X_n\} \qquad \text{(ii) } Z = \mathrm{Max}\{X_1, ..., X_n\}$$

[*Hint: find the ranges of Y and Z, and consider* $\mathrm{P}[\ Y = 1\ ]$, $\mathrm{P}[\ Z = 0\ ]$]

8. Consider an event E in sample space Ω, and consider the *indicator function*, I_E, defined for $\omega \in \Omega$ by

$$I_E(\omega) = \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E \end{cases}$$

Show that I_E defines a *Bernoulli* random variable.

Show furthermore that **any** discrete random variable can be expressed as a linear combination of indicator random variables.

9. The *Poisson distribution* is a probability model that is commonly used to specify probabilities for the outcomes of counting experiments. If X is a discrete random variable taking values on range $\mathbb{X} = \{0, 1, 2, ...\}$ with probability mass function f_X given by

$$f_X(x) = \mathrm{P}[\, X = x \,] = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, ...$$

for parameter $\lambda > 0$, then X has a Poisson distribution with parameter λ, or $X \sim Poisson(\lambda)$.

(i) Verify that the Poisson distribution is a valid discrete probability distribution.

(ii) Suppose that Z is a discrete random variable with $\mathrm{P}[\, Z = 0 \,] = 0$, but otherwise with $\mathrm{P}[\, Z = z \,] \propto \mathrm{P}[\, X = z \,]$ for $z \in \mathbb{R}$. Find the probability mass function of Z.

(iii) Suppose now that $X_1 \sim Poisson(\lambda_1)$ and $X_2 \sim Poisson(\lambda_2)$ are **independent** random variables, so that

$$\mathrm{P}[\, (X_1 = x_1) \cap (X_2 = x_2) \,] = \mathrm{P}[\, (X_1 = x_1) \,]\, \mathrm{P}[\, (X_2 = x_2) \,] \qquad \text{for all pairs } (x_1, x_2)$$

If random variable Y is defined by $Y = X_1 + X_2$, prove that $Y \sim Poisson(\lambda_1 + \lambda_2)$.

Hint: consider the partition

$$(Y = y) = \bigcup_{x_1=0}^{\infty} ((X_1 = x_1) \cap (X_2 = y - x_1)) \qquad \text{for } y = 0, 1, 2, ...$$

10. For a discrete random variable X with range $\mathbb{X}$ and probability mass function f_X, define the **probability generating function** (or pgf) of X, G_X, as a power series in t by

$$G_X(t) = \sum_{x \in \mathbb{X}} f_X(x)\, t^x$$

for values of t where the sum is convergent. The pgf is a function of t that is specific to each mass function f_X; each f_X has a unique corresponding G_X.

(i) Find the pgf of X if $X \sim Poisson(\lambda)$.

(ii) Prove that if X_1 and X_2 are independent random variables, and $Y = X_1 + X_2$, then

$$G_Y(t) = G_{X_1}(t)\, G_{X_2}(t)$$