M1S : EXERCISES 4

COMBINATORICS : HYPERGEOMETRIC PROBABILITIES, PARTITIONING AND OCCUPANCY PROBLEMS

1. The *binomial expansion* is used to express $(a+b)^n$ in power series form:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Use the binomial expansion to prove the following identities:

For non-negative integers m and n, and $0 \leq k \leq m+n$,

(i) $$1 - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$$

(ii) $$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

Hypergeometric Probabilities

2. The National Lottery each week randomly produces 6 winning numbers, drawn without replacement from the integers 1,2,...,49.

(a) If you have 1 entry ticket (i.e. one choice of 6 numbers), what is your chance of having the winning sequence of 6?

(b) A prize of £10 is won if exactly 3 of your numbers are in the winning sequence. What is the probability of such a win if you have 1 entry ticket?

(c) What is the probability that, on any week, the ticket contains no winning numbers ?

(d) Assuming that draws on successive weeks are independent, find an expression in x for the probability that the tickets purchased purchased contain no winning numbers for x successive weeks, where $x = 1, 2, 3.....$

3. The game of *Keno* is a lottery-type game which consists of cards numbered 1,2,...,80, twenty of which are to be drawn as winning cards. A player attempts to win at the game by nominating n cards before the draw is made, where n is any number not greater than fifteen. The amount won depends on the number of nominated cards that are subsequently drawn as winning cards.

What is the probability that no nominated cards are drawn as winning cards if

(i) $n = 5$ (ii) $n = 10$ (iii) $n = 15$

What is the general formula for the probability of obtaining r winning cards ?

4. A committee of $n = 5$ students is to be selected, supposedly at random, from a class of $N = 200$ comprising $R = 120$ males and $N - R = 80$ females.

Comment (using probability reasoning only ...) on the evidence for sex-bias in the selection process if the committee comprises $r = 5$ males, and no females.

[*Evaluate the probability of the observed event assuming that the selection process is truly random, that is, that all selections of* $n = 5$ *from* $N = 200$ *are equally likely.*]

Combinatorial Partitions And Occupancy Problems

5. A bridge hand consists of 13 cards from a normal deck of 52.

(a) How many different bridge hands are there?

(b) What is the probability that a randomly chosen bridge hand contains no aces, kings, queens or jacks?

6. A class comprises 30 pupils. What is the probability that, among the twelve months, six contain two of the birthdays, and six contain three of the birthdays of the pupils ?

7. A group of $2n$ females and $2n$ males is randomly divided into two subgroups. What is the probability that each subgroup contains n females and n males ?

8. Two hands of thirteen cards are dealt (without replacement) from an ordinary pack. What is the probability that one hand contains exactly n Hearts, and the other contains exactly m Hearts ?

Now suppose that four hands of thirteen cards are dealt (without replacement) from an ordinary pack. What is the probability that the four hands contain r_1, r_2, r_3, r_4 Hearts respectively, for non-negative integers satisfying $r_1 + r_2 + r_3 + r_4 = 13$?

9. n people, including yourself and a friend, are seated at random in a row of n chairs. What is the probability that you sit next to your friend?

10. If n balls are placed at random into n boxes, what is the probability that precisely one box remains empty ?

11. Each of n sticks is broken into two parts, long and short, and a new set of n sticks formed by pairing and joining the $2n$ parts at random. What is the probability that

(i) each stick is paired and rejoined into its original form that is, there is a match between the rejoined long and short parts for all n sticks.

(ii) each of the n long parts are rejoined with a short part.