M1S : EXERCISE SHEET 3
CONDITIONAL PROBABILITY AND THE PROBABILITY THEOREMS

1. Given an event F in sample space Ω with $\mathrm{P}(F) > 0$, the conditional probability operator, defined on the events of Ω, satisfies the probability axioms (I), (II) and (III), that is, for events $E_1, E_2 \subseteq \Omega$,

$$\begin{array}{ll} \text{(I)} & 0 \leq \mathrm{P}(E_1 \mid F) \leq 1 \\ \text{(II)} & \mathrm{P}(\Omega \mid F) = 1 \\ \text{(III)} & E_1 \cap E_2 = \emptyset \implies \mathrm{P}(E_1 \cup E_2 \mid F) = \mathrm{P}(E_1 \mid F) + \mathrm{P}(E_2 \mid F) \end{array}$$

Verify these results in the Relative Frequency interpretation of probability.

2. Suppose that E, F are **independent** events.
(a) Show that E', F and E', F' are also independent pairs of events.
(b) If events E, F are also disjoint, what can you say about $\mathrm{P}(E)$ and $\mathrm{P}(F)$?

3. *The General Multiplication Rule*

Consider the space module system represented in EXERCISES 1, Q. 6, and define events A, A_1, A_2 etc corresponding to the stages A, A_1, A_2 etc functioning.

(a) If the probability of A_1 functioning is 0.95, and that of A_2 functioning (if required) is 0.90, and A_1 and A_2 function or fail independently, what is the probability that A functions ?

(b) If the events B_1, B_2 and B_3 are mutually independent, and each has probability 0.20 of failing, what is the probability that B functions ?

(c) Assuming that the internal functioning of the five stages are mutually independent, and that each of the last three stages has probability 0.05 of failing, calculate the probability that the entire system functions.

4. *The Monty Hall Game Show Problem: An application of The Theorem of Total Probability and Bayes Theorem*

In a TV Game show, a contestant selects one of three doors; behind one of the doors there is a prize, and behind the other two there are no prizes. After the contestant selects a door, the game-show host opens one of the remaining doors, and reveals that there is no prize behind it. The host then asks the contestant whether they want to SWITCH their choice to the other unopened door, or STICK to their original choice.

Is it probabilistically advantageous for the contestant to SWITCH doors, or is the probability of winning the prize the same whether they STICK or SWITCH ? (Assume that the host selects a door to open, from those available, with equal probability).

[Solution of this problem requires a simple application of conditional probability and Bayes Theorem, and is meant to illustrate that the correct answer is occasionally counter-intuitive. For a detailed solution, see, for example, *www.io.com/~kmellis/monty.html*.]

5. *The Prisoner's Dilemma*

Three prisoners A, B, C are in solitary confinement under sentence of death, but each knows that one of them, chosen at random with equal probability, is to be pardoned. Prisoner A begs the governor to tell him whether he, A, is to be pardoned or executed. The governor refuses to answer this, but he does say that B is to be executed. The governor thinks that he isn't giving useful information, as A knows that at least one of B and C must die.

A suddenly feels much happier, as he believes his chances of being pardoned have *risen* from 1/3 to 1/2. The governor, who, if A were actually to be pardoned, would be equally likely to give C's name rather than B's, is mystified by A's euphoria. Who is correct?

[Hint: Let A, B, C be the events that A, B or C respectively are pardoned. Then A, B, C partition Ω. Now let G_{AB} be the event that the governor tells A that B is to be executed. You want $\mathrm{P}(A \mid G_{AB})$, so consider the three conditional probabilities of G_{AB} given A, B and C respctively, and then use Bayes Theorem.]

What C should feel if he overhears the governor's reply, but assumes that the question was asked by one of the warders? (consider the event G_{WB} that the governor tells a warder that B is to be executed).

6. A crime has been committed and a suspect is being held by police. He is either guilty, G, or not, G', and the probability of his being guilty on the basis of current evidence is $\mathrm{P}(\,G\,) = p$, say. Forensic evidence is now produced which shows that the criminal must have a property, A, which occurs in a proportion, π, of the general population. Suppose that if the suspect is innocent he can be treated as a member of the general population, so that $\mathrm{P}(\,A \mid G'\,) = \pi$.

The suspect is now interrogated and found to have property A. Show that the odds on his guilt have now risen from $\lambda_0 = p/(1-p)$ to $\lambda_1 = \lambda_0/\pi$.

Recall that the odds on an event E are defined to be the ratio $\mathrm{P}(\,E\,)/\mathrm{P}(\,E'\,)$, the odds-against E are $\mathrm{P}(\,E'\,)/\mathrm{P}(\,E\,)$.

7. A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive when applied to a non-sufferer. It is estimated that 0.5 % of the population are sufferers. Suppose that the test is now administered to a person about whom we have no relevant information relating to the disease (apart from the fact that he/she comes from this population). Calculate the following probabilities:

(a) that the test result will be positive;

(b) that, given a positive result, the person is a sufferer;

(c) that, given a negative result, the person is a non-sufferer;

(d) that the person will be misclassified.

8. Athletes are routinely tested for the use of performance-enhancing drugs. When a test is to be carried out, the athlete provides two blood samples, the first of which is then tested. If this test is positive, indicating that drugs are present, the second sample is tested, and if the second test is also positive, then the athlete has failed the test.

Suppose that an athlete is selected at random, and two blood samples (regarded as identical) are obtained. Let events T_1 and T_2 correspond respectively to the events that first and second samples test positive, and let C be the event that drugs are actually present in the samples. Suppose also that the test used is quite accurate, in that it correctly indicates the *presence* of drugs in 99.5% of tests, and correctly indicates the *absence* of drugs in 98% of tests.

It is estimated that only 1 athlete in 1000 gives samples in which drugs are actually present

If it is assumed the results of the two tests are *conditionally independent* given the presence or absence of drugs in the samples, give expressions for, and evaluate

(a) the probability that the first test is positive

(b) the conditional probability that drugs are actually present in the sample, given that the first test is positive.

(c) the probability that both tests are positive, so that the athlete fails the test

(d) the conditional probability that drugs are actually present in the sample, given that both tests are positive.

For events A, B and C, with $\mathrm{P}(C) > 0$, A and B are *conditionally independent* given C if

$$\mathrm{P}(\,A \cap B \mid C\,) = \mathrm{P}(\,A \mid C\,)\mathrm{P}(\,B \mid C\,)$$