M1S : EXERCISE SHEET 2

ELEMENTARY PROBABILITY

EXERCISES

1. A person selected from a population and subjected to screening-for-a particular-disease is either a sufferer (S) or not (S'). The screening will either result in a positive test (T), or a negative (T'). If the result of the test is used to classify the person, write down the event "person misclassified".

Suppose now that an X-ray is also taken and either gives a positive (X), or a negative (X') indication, and also a doctor carries out an examination resulting in a positive (D), or negative (D') assessment. If a person is classified as a sufferer if and only if the doctor and at least one of the tests points in this direction, represent the event "correct classification made".

2. Given two events $E, F \subseteq \Omega$, prove that the probability of *one and only one* of them occurring is

$$\mathrm{P}(E) + \mathrm{P}(F) - 2\mathrm{P}(E \cap F).$$

3. Consider the following statements, which are claimed to be true for events E_1, E_2 in a sample space Ω:

$$\begin{array}{lll}
(a) & \mathrm{P}(E_1) = 0 & \Longrightarrow \mathrm{P}(E_1 \cup E_2) = 0 \\
(b) & \mathrm{P}(E_1) = \mathrm{P}(E_2') & \Longrightarrow E_1' = E_2 \\
(c) & E_1 \subseteq E_2 \text{ and } \mathrm{P}(E_1) = P(E_2') & \Longrightarrow P(E_1) \leq 1/2 \\
(d) & \mathrm{P}(E_1') = x_1, P(E_2') = x_2 & \Longrightarrow P(E_1 \cup E_2) \geq 1 - x_1 - x_2
\end{array}$$

In each case, either prove that the statement is true for all Ω, E_1, E_2, or provide a specific counter-example to show that there exists Ω, E_1, E_2 for which it is false

4. Given two events $E, F \subseteq \Omega$, prove that

$$\begin{array}{ll}
(a) & \mathrm{P}(E' \cap F) = \mathrm{P}(F) - \mathrm{P}(E \cap F) \\
(b) & \mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F) \\
(c) & E \subseteq F \Longrightarrow \mathrm{P}(E) \leq \mathrm{P}(F) \\
(d) & \mathrm{P}(E \cap F) \geq \mathrm{P}(E) + \mathrm{P}(F) - 1
\end{array}$$

[(d) is known as *Bonferroni's Inequality.*]

5. Suppose that E and F are events such that $\mathrm{P}(E) = x, \mathrm{P}(F) = y$ and $\mathrm{P}(E \cap F) = z$. Express the following terms of x, y and z:

$$\begin{array}{ll}
(a) & P(E' \cup F') \\
(b) & P(E' \cap F) \\
(c) & P(E' \cup F) \\
(d) & P(E' \cap F').
\end{array}$$

6. Show that the first probability axiom can be relaxed to:

$$\text{(I)}^* : \mathrm{P}(E) \geq 0 \text{ for all } E \subseteq \Omega,$$

that is, given that (I)*, (II) and (III) hold, prove that $\mathrm{P}(E) \leq 1$ for all $E \subseteq \Omega$.

7. Use induction to prove that, for any events $E_1, E_2, ..., E_n$ in Ω:

$$(a) \quad \mathrm{P}(E_1 \cup E_2 \cup ... \cup E_n) \quad \leq \quad P(E_1) \ + \ ... \ + \ P(E_n) \quad (\textit{Boole's Inequality})$$

$$(b) \quad \mathrm{P}(E_1 \cup E_2 \cup ... \cup E_n) \quad = \quad \sum_i \mathrm{P}(E_i) \ - \sum_i \sum_j \mathrm{P}(E_i \cap E_j) \ + \ \sum_i \sum_j \sum_k \mathrm{P}(E_i \cap E_j \cap E_k) \ - \ ... \ +$$

$$(-1)^{n-1} \ \mathrm{P}(E_1 \cap E_2 \cap ... \cap E_n)$$

In (b), summations are over *distinct* sets of subscripts, each counted once only. For example, if $n = 3$, the result would correspond to

$$\mathrm{P}(E_1 \cup E_2 \cup E_3) \ = \ \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) \ - \ \mathrm{P}(E_1 \cap E_2) - \mathrm{P}(E_1 \cap E_3) - \mathrm{P}(E_2 \cap E_3) \ + \ \mathrm{P}(E_1 \cap E_2 \cap E_3).$$