M1S : EXERCISE SHEET 2

ELEMENTARY PROBABILITY

EXERCISES

1. A person selected from a population and subjected to screening-for-a particular-disease is either a sufferer (S) or not (S'). The screening will either result in a positive test (T), or a negative (T'). If the result of the test is used to classify the person, write down the event "person misclassified".

Suppose now that an X-ray is also taken and either gives a positive (X), or a negative (X') indication, and also a doctor carries out an examination resulting in a positive (D), or negative (D') assessment. If a person is classified as a sufferer if and only if the doctor and at least one of the tests points in this direction, represent the event "correct classification made".

2. Given two events $E, F \subseteq \Omega$, prove that the probability of *one and only one* of them occurring is

$$\mathrm{P}(E) + \mathrm{P}(F) - 2\mathrm{P}(E \cap F).$$

3. Consider the following statements, which are claimed to be true for events E_1, E_2 in a sample space Ω:

$$(a) \quad \mathrm{P}(E_1) = 0 \implies \mathrm{P}(E_1 \cup E_2) = 0$$

$$(b) \quad \mathrm{P}(E_1) = \mathrm{P}(E_2') \implies E_1' = E_2$$

$$(c) \quad E_1 \subseteq E_2 \text{ and } \mathrm{P}(E_1) = P(E_2') \implies P(E_1) \leq 1/2$$

$$(d) \quad \mathrm{P}(E_1') = x_1, P(E_2') = x_2 \implies P(E_1 \cup E_2) \geq 1 - x_1 - x_2$$

In each case, either prove that the statement is true for all Ω, E_1, E_2, or provide a specific counter-example to show that there exists Ω, E_1, E_2 for which it is false

4. Given two events $E, F \subseteq \Omega$, prove that

$$(a) \quad \mathrm{P}(E' \cap F) = \mathrm{P}(F) - \mathrm{P}(E \cap F)$$

$$(b) \quad \mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F)$$

$$(c) \quad E \subseteq F \implies \mathrm{P}(E) \leq \mathrm{P}(F)$$

$$(d) \quad \mathrm{P}(E \cap F) \geq \mathrm{P}(E) + \mathrm{P}(F) - 1$$

[(d) is known as *Bonferroni's Inequality.*]

5. Suppose that E and F are events such that $\mathrm{P}(E) = x, \mathrm{P}(F) = y$ and $\mathrm{P}(E \cap F) = z$. Express the following terms of x, y and z:

$$(a) \quad P(E' \cup F')$$

$$(b) \quad P(E' \cap F)$$

$$(c) \quad P(E' \cup F)$$

$$(d) \quad P(E' \cap F').$$

6. Show that the first probability axiom can be relaxed to:

$$(\mathrm{I})^* : \mathrm{P}(E) \geq 0 \text{ for all } E \subseteq \Omega,$$

that is, given that $(\mathrm{I})^*$, (II) and (III) hold, prove that $\mathrm{P}(E) \leq 1$ for all $E \subseteq \Omega$.

7. Use induction to prove that, for any events $E_1, E_2, ..., E_n$ in Ω:

(a) $\mathrm{P}(E_1 \cup E_2 \cup ... \cup E_n) \leq P(E_1) + ... + P(E_n)$ (*Boole's Inequality*)

(b) $$\mathrm{P}(E_1 \cup E_2 \cup ... \cup E_n) = \sum_i \mathrm{P}(E_i) - \sum_i \sum_j \mathrm{P}(E_i \cap E_j) + \sum_i \sum_j \sum_k \mathrm{P}(E_i \cap E_j \cap E_k) - ... + (-1)^{n-1}\,\mathrm{P}(E_1 \cap E_2 \cap ... \cap E_n)$$

In (b), summations are over *distinct* sets of subscripts, each counted once only. For example, if $n = 3$, the result would correspond to

$$\mathrm{P}(E_1 \cup E_2 \cup E_3) = \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) - \mathrm{P}(E_1 \cap E_2) - \mathrm{P}(E_1 \cap E_3) - \mathrm{P}(E_2 \cap E_3) + \mathrm{P}(E_1 \cap E_2 \cap E_3).$$