M1S : ASSESSED COURSEWORK 3

Deadline : Thursday, 8th March

A **continuous** random variable X is a random variable that takes values on range $\mathbb{X}$ that is an interval (or union of intervals) of $\mathbb{R}$ containing an *uncountable* number of values x; the probability distribution of X is specified by using either the *continuous cumulative distribution function* (cdf), denoted F_X, that is defined by

$$F_X(x) = \mathrm{P}\left[X \leq x\right] \qquad\qquad x \in \mathbb{R}$$

or by the *probability density function* (pdf), denoted f_X, that is defined **implicitly** by the integral

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

or **explicitly** as the derivative

$$f_X(x) = \frac{d}{dt}\left\{F_X(t)\right\}_{t=x}$$

(here t is a dummy variable; the right hand side is merely the first derivative of F_X with respect to x).

It can be shown that the pdf **must** satisfy the two conditions

$$\text{(i)} \quad f_X(x) \geq 0 \qquad\qquad \text{(ii)} \quad \int_{\mathbb{X}} f_X(x)\, dx = 1$$

and by convention, we define

$$f_X(x) = 0 \qquad\qquad x \notin \mathbb{X}$$

1. (i) The function $f(x)$ defined by

$$f(x) = -\ln x$$

is to be used as the pdf for a continuous random variable X. Find the range $\mathbb{X}$ on which X must be defined in order to make this a valid pdf specification, and find the corresponding continuous cdf.

[*5 MARKS*]

(ii) Consider continuous rv X with range $\mathbb{X} \equiv \mathbb{R}^+ = \{x : x > 0\}$. Let the *hazard* function, denoted h_X, be defined by

$$h_X(x) = \frac{f_X(x)}{1 - F_X(x)} \qquad\qquad x > 0$$

By viewing this definition as a separable, first order differential equation in $y = F_X(x)$, show that

$$F_X(x) = 1 - \exp\left\{-\int_0^x h_X(t)\, dt\right\}$$

[*5 MARKS*]

2 The Poisson process is a statistical model for events occurring in continuous time; if X_t is the discrete random variable counting the **number** of events that occur in an interval $[0, t)$, then we have that

$$\mathrm{P}\left[X_t = n\right] = \frac{e^{-\lambda t}\left(\lambda t\right)^n}{n!} \qquad n = 0, 1, 2, ...$$

so that $X_t \sim Poisson(\lambda t)$, for some rate parameter $\lambda > 0$.

Let **continuous** random variable X be defined as the **time** at which the nth event occurs in this Poisson process. Prove that the pdf of X is given by

$$f_X(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} \qquad x > 0$$

[*10 MARKS*]

Hint: Consider first the cdf of X, denoted F_X; we have that for time $x > 0$

$$\begin{aligned} F_X(x) &= \mathrm{P}\left[X \leq x\right] = 1 - \mathrm{P}\left[X > x\right] \\ &= 1 - \mathrm{P}\left[n\text{th event occurs } \textit{later} \text{ than time } x\right] \\ &= 1 - \mathrm{P}\left[\textit{fewer} \text{ than } n \text{ events occur in } [0, x)\right] \end{aligned}$$

Then use the Poisson process/Poisson model assumptions, and then differentiate.