M1S : EXERCISE SHEET 7 : SOLUTIONS

ASSESSED

(a) (i) The Poisson mass function is a valid specification as, first, it is non-negative for all $x \in \mathbb{R}$, and secondly as

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda}\ e^{\lambda} = 1$$

by a standard power series result.

[2 MARKS]

(ii) For $z = 1, 2, ...$, write $f_Z(z) = \mathrm{P}[\ Z = z\ ] = c\mathrm{P}[\ X = z\ ]$. Then

$$\sum_{z=1}^{\infty} f_Z(z) = \sum_{z=1}^{\infty} c\ \mathrm{P}[\ X = z\ ] = c\sum_{z=1}^{\infty} f_X(z) = c\ \mathrm{P}[\ X > 0\ ] = c\ \ (1 - \mathrm{P}[\ X = 0\ ]) = c(1 - e^{-\lambda})$$

Hence $c = 1/\left(1 - e^{-\lambda}\right)$, and so

$$f_Z(z) = \frac{e^{-\lambda}\lambda^z}{(1 - e^{-\lambda})\, z!} \qquad z = 1, 2, ...$$

and zero otherwise.

[3 MARKS]

(iii) Using the partition, and/or the Theorem of Total Probability, for $y = 0, 1, 2, ...$,

$$\begin{aligned} f_Y(y) = \mathrm{P}[\ Y = y\ ] &= \sum_{x_1=0}^{\infty} \mathrm{P}[\ (X_1 = x_1) \cap (X_2 = y - x_1)\ ] = \sum_{x_1=0}^{y} \mathrm{P}[\ X_1 = x_1\ ]\mathrm{P}[\ X_2 = y - x_1\ ] \\ &= \sum_{x_1=0}^{y} \frac{e^{-\lambda_1}\lambda_1^{x_1}}{x_1!}\frac{e^{-\lambda_2}\lambda_2^{y-x_1}}{(y-x_1)!} = e^{-(\lambda_1+\lambda_2)}\sum_{x_1=0}^{y}\frac{1}{x_1!(y-x_1)!}\lambda_1^{x_1}\lambda_2^{y-x_1} \\ &= \frac{e^{-(\lambda_1+\lambda_2)}}{y!}\sum_{x_1=0}^{y}\frac{y!}{x_1!(y-x_1)!}\lambda_1^{x_1}\lambda_2^{y-x_1} = \frac{e^{-(\lambda_1+\lambda_2)}}{y!}(\lambda_1+\lambda_2)^y \end{aligned}$$

where the first line follows from independence of X_1 and X_2. Hence $Y \sim Poisson(\lambda_1 + \lambda_2)$.

[5 MARKS]

(b)(i) If $X \sim Poisson(\lambda)$, then

$$G_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}t^x = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda t)^x}{x!} = e^{-\lambda}\ e^{\lambda t} = e^{\lambda(t-1)}$$

[4 MARKS]

(ii) If X_1 and X_2 are independent, and $Y = X_1 + X_2$, then by using the argument from (iii) above,

$$\begin{aligned} G_Y(t) &= \sum_{y=-\infty}^{\infty} f_Y(y)t^y = \sum_{y=-\infty}^{\infty}\left\{\sum_{x_1=-\infty}^{\infty} f_{X_1}(x_1)f_{X_2}(y-x_1)\right\}t^y \\ &= \sum_{x_1=-\infty}^{\infty}\sum_{x_2=-\infty}^{\infty} f_{X_1}(x_1)f_{X_2}(x_2)t^{x_1+x_2} \qquad \text{changing variables to } x_2 = y - x_1 \\ &= \left\{\sum_{x_1=-\infty}^{\infty} f_{X_1}(x_1)t^{x_1}\right\}\left\{\sum_{x_2=-\infty}^{\infty} f_{X_2}(x_2)t^{x_2}\right\} = G_{X_1}(t)\ G_{X_2}(t) \end{aligned}$$

[6 MARKS]

EXERCISES

1. Need to show f_X non-negative, and sums to one over the range of X. Sum of geometric progression gives result, that is,

$$\sum_{x=0}^{\infty} f_X(x) = \sum_{x=0}^{\infty} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^x = \frac{1}{1+\lambda}\left(1-\frac{\lambda}{1+\lambda}\right)^{-1} = 1$$

Distribution function F_X given, for $x = 0, 1, ...$ by

$$F_X(x) = \sum_{i=0}^{x} f_X(i) = \sum_{i=0}^{x} \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^i = (1-\theta)\sum_{i=0}^{x}\theta^i = (1-\theta)\frac{1-\theta^{x+1}}{1-\theta} = 1-\theta^{x+1}$$

where $\theta = \dfrac{\lambda}{1+\lambda}$.

2. $Y = X - n \implies \mathbb{Y} = \{0, 1, 2, ...\}$ and

$$f_Y(y) = \mathrm{P}[Y = y] = \mathrm{P}[X - n = y] = \mathrm{P}[X = y + n] = f_X(y+n) = \binom{n+y-1}{n-1}\theta^n(1-\theta)^y$$

3. $X \sim \mathit{Hypergeometric}(N, R, n)$, so for $x \in \{\mathrm{Max}(0, n-N+R), ..., \mathrm{Min}(n, R)\}$,

$$\begin{aligned} f_X(x) &= \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{N}{n}} = \frac{R!}{x!(R-x)!}\frac{(N-R)!}{(n-x)!(N-R-n+x)!}\frac{n!(N-n)!}{N!} \\ &= \binom{n}{x}\frac{R!}{(R-x)!}\frac{(N-R)!}{(N-R-n+x)!}\frac{(N-n)!}{N!} \approx \binom{n}{x}R^x(N-R)^{n-x}\frac{1}{N^n} = \binom{n}{x}\theta^x(1-\theta)^{n-x} \end{aligned}$$

where $\theta = R/N$. Hence for N and R large, with $\theta = R/N$ fixed, the hypergeometric distribution is tends to the binomial distribution, and thus sampling without replacement from a large population is approximately equivalent to sampling with replacement.

4. Let $U =$"Number of Heads", then $X = U - (n - U) = 2U - n$. Thus

$$\mathbb{X} = \{-n, -n+2, -n+4, ..., n-4, n-2, n\}$$

and

$$f_X(x) = \mathrm{P}[X = x] = \mathrm{P}[2U - n = x] = \mathrm{P}[U = (x+n)/2] = f_U((x+n)/2))$$

where $U \sim \mathit{Binomial}(n, 1/2)$.

5. If $X \sim \mathit{Geometric}(\theta)$, then

$$f_X(x) = (1-\theta)^{x-1}\theta \qquad F_X(x) = 1-(1-\theta)^x$$

for $x \in \{1, 2, 3...\}$. Thus $\mathrm{P}[\ X > n\ ] = (1-\theta)^n$, and hence

$$\mathrm{P}[\ X = n+k \mid X > n\ ] = \frac{\mathrm{P}[\ X = n+k, X > n\ ]}{\mathrm{P}[\ X > n\ ]} = \frac{\mathrm{P}[\ X = n+k\ ]}{\mathrm{P}[\ X > n\ ]} = \frac{(1-\theta)^{n+k-1}\theta}{(1-\theta)^n} = (1-\theta)^{k-1}\theta = \mathrm{P}[\ X = k\ ]$$

6. Need $f(x)$ non-negative and convergent;

(i) always non-negative and convergent; need $k = 1$

(ii) always non-negative and convergent if $\alpha < -1$; no closed form for k.

7.(i) $X = \text{Min}\{X_1, ..., X_n\}$, so $\mathbb{X} = \{0, 1\}$.

$$\text{P}[\ X = 1\ ] = \text{P}[\ \text{Min}\{X_1, ..., X_n\} = 1] = \text{P}[\ X_1 = 1, X_2 = 1, ..., X_n = 1\ ] = \theta^n \qquad \text{P}[\ X = 0\ ] = 1 - \theta^n$$

Hence

$$f_X(x) = \begin{cases} 1 - \theta^n & x = 0 \\ \theta^n & x = 1 \end{cases}$$

(ii) $X = \text{Max}\{X_1, ..., X_n\}$, so $\mathbb{X} = \{0, 1\}$.

$$\text{P}[\ X = 0\ ] = \text{P}[\ \text{Max}\{X_1, ..., X_n\} = 0] = \text{P}[\ X_1 = 0, X_2 = 0, ..., X_n = 0\ ] = (1-\theta)^n \qquad \text{P}[\ X = 0\ ] = 1 - (1-\theta)^n$$

Hence

$$f_X(x) = \begin{cases} (1 - \theta)^n & x = 0 \\ 1 - (1 - \theta)^n & x = 1 \end{cases}$$

8. $I_E : \Omega \longrightarrow \{0, 1\}$ and

$$\text{P}[\ I_E(\omega) = 1\ ] \equiv \text{P}[\ \omega \in E\ ] = \text{P}(E) = \theta$$

and hence I_E is a Bernoulli random variable with parameter θ.

If X is discrete, can write $\mathbb{X} = \{x_1, x_2, ...\}$, and hence

$$X = \sum_{i=1}^{\infty} I_{E_i} x_i$$

where

$$E_i = \{\omega : X(\omega) = x_i\}$$