M1S : EXERCISE SHEET 4 : SOLUTIONS

1. (a) Special case of the Binomial Expansion of $(a+b)^n$ with $a = 1, b = -1$.

(b) $(x+1)^{m+n} = (x+1)^m(x+1)^n$, and

$$
\begin{aligned}
L.H.S. \quad : \quad (x+1)^{m+n} \quad &= \sum_{k=0}^{m+n} \binom{m+n}{k} x^k \\
R.H.S. \quad : \quad (x+1)^m(x+1)^n \quad &= \left\{ \sum_{i=0}^{m} \binom{m}{i} x^i \right\} \left\{ \sum_{j=0}^{n} \binom{n}{j} x^j \right\} \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} x^{i+j} \\
&= \sum_{k=0}^{m+n} \left\{ \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} \right\} x^k
\end{aligned}
$$

putting $i + j = k$, and re-arranging summations. Result follows by equating left and right hand sides and comparing coefficients of x_k.

2. Hypergeometric Probabilities with $N = 49$, $R = 6$ (winning balls), and $n = 6$ (drawn balls).

$$
\text{(a)} \quad r = 6 \qquad \text{Probability} = \frac{\binom{6}{6}\binom{43}{0}}{\binom{49}{6}} = \frac{1}{13983816} \approx 7.15 \times 10^{-8}
$$

$$
\text{(b)} \quad r = 3 \qquad \text{Probability} = \frac{\binom{6}{3}\binom{43}{3}}{\binom{49}{6}} \approx 0.0177
$$

3. Each of 10 digits available on each draw, so total number of possible draws is 10^k (by the Multiplication Theorem).

(a) Let $A_r = \{$Sequences where all digits $\leq r\}$. For sequences in the set A_r, there are $r+1$ digits available for each draw, so the number of sequences in A_r is $(r+1)^k$, and hence for $r = 0, 1.2, ..., 9$,

$$
\mathrm{P}(A_r) = \left(\frac{r+1}{10} \right)^k
$$

(b) Let $B_r = \{$Sequences where largest digit is $r\}$. Then $A_r = A_{r-1} \cup B_r$, where A_{r-1} and B_r are disjoint sets. Hence the number of sequences in the set B_r is given by $(r+1)^k - r^k$, and

$$
\mathrm{P}(B_r) = \frac{(r+1)^k - r^k}{10^k}
$$

4. Let $n =$ Total number of hands (number of ways of choosing five cards from 52), so $n = \binom{52}{5}$. Now need to calculate n_E for the events of interest.

(a) ROYAL FLUSH: $n_E = 4$ (select suit) , so

$$\mathrm{P}(E) = \frac{n_E}{n} = \frac{4}{\binom{52}{5}} \approx 1.5 \times 10^{-6}$$

(b) FOUR OF A KIND ($xxxxy$):

$$n_E = \binom{13}{1}\binom{4}{4} \times \binom{12}{1}\binom{4}{1} = 13 \times 48$$

(choose a denomination for x, then four cards in that denomination, then a denomination for y, then one card in that denomination). Hence

$$\mathrm{P}(E) = \frac{n_E}{n} = \frac{\binom{13}{1}\binom{4}{4} \times \binom{12}{1}\binom{4}{1}}{\binom{52}{5}} \approx 2.4 \times 10^{-4}$$

(b) TWO PAIRS ($xxyyz$):

$$n_E = \binom{13}{2}\binom{4}{2}\binom{4}{2} \times \binom{11}{1}\binom{4}{1} = 13 \times 48$$

(choose a pair of *different* denominations for (x, y), then two cards from four in the x denomination and then two cards from four in the y denomination, then a denomination for z, then one card in that denomination). Hence

$$\mathrm{P}(E) = \frac{n_E}{n} = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2} \times \binom{11}{1}\binom{4}{1}}{\binom{52}{5}} \approx 0.0475.$$

5. (a) Choose 13 cards from 52 $\Longrightarrow n = \binom{52}{13}$

(b) Choose 13 cards from 36 $\Longrightarrow n_E = \binom{36}{13} \Longrightarrow \mathrm{P}(E) = \dfrac{n_E}{n} \approx 0.003$

6. Total number of seating arrangements is $n!$. Of these, you and friend sit together in

$$(n-1) \times 2! \times (n-2)!$$

arrangements (choose a pair of adjacent seats for you and friend from $n-1$ pairs available, arrange yourselves in one of $2!$ ways, and then arrange remaining $n-2$ people in remaining seats). Hence probability is

$$\frac{(n-1) \times 2! \times (n-2)!}{n!} = \frac{2}{n}$$

TUTORIAL

(a) Birthday problem studied in lectures;

$$\mathrm{P}(E) = \frac{n_E}{n} = \frac{365 \times 364 \times \ldots \times (365 - N + 1)}{365 \times 365 \times \ldots \times 365} = \frac{365!}{(365-N)!}\frac{1}{365^N} = \frac{(365)_N}{365^N}$$

(b) Occupancy problem; allocate six "balls" to six "cells" with no cells empty, that is, with one ball in each cell.

$$\mathrm{P}(E) = \frac{6!}{6^6}$$

(c) Poker hands; denominator is always $n = \begin{pmatrix} 52 \\ 5 \end{pmatrix}$.

Hand		n_E	Prob.
ONE PAIR	*xxabc*	$\binom{13}{1}\binom{4}{2} \times \binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$	0.42
TWO PAIRS	*xxyya*	$\binom{13}{2}\binom{4}{2}\binom{4}{2} \times \binom{11}{1}\binom{4}{1}$	0.048
THREE OF A KIND	*xxxab*	$\binom{13}{1}\binom{4}{3} \times \binom{12}{2}\binom{4}{1}\binom{4}{1}$	0.021
STRAIGHT	Run of five cards	$\binom{10}{1}\binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}$	0.0039
FLUSH	Five cards in same suit	$\binom{13}{5}\binom{4}{1} - \binom{10}{1}\binom{4}{1}$	0.0020
FULL HOUSE	*xxxyy*	$\binom{13}{1}\binom{4}{3} \times \binom{12}{1}\binom{4}{2}$	0.0014
FOUR OF A KIND	*xxxxa*	$\binom{13}{1}\binom{4}{4} \times \binom{12}{1}\binom{4}{1}$	0.00024
STRAIGHT FLUSH	Run of five cards in same suit	$10\binom{4}{1} - \binom{4}{1}$	0.000014
ROYAL FLUSH	AKQJ10 in any suit	$1\binom{4}{1}$	0.0000015

For STRAIGHT and FLUSH, must remember to subtract the STRAIGHT FLUSHes (which are counted separately); similarly, for STRAIGHT FLUSH, must remember to subtract ROYAL FLUSHes.