M1S : EXERCISE SHEET 2 : SOLUTIONS

ASSESSED

(a) (i) Let $\omega \in L.H.S.$; then

$$\omega \in \left(\bigcup_{i=1}^{n} E_i\right)' \Longrightarrow \omega \notin \bigcup_{i=1}^{n} E_i \Longrightarrow \omega \in E_i' \text{ for all } i \Longrightarrow \omega \in \bigcap_{i=1}^{n} E_i'$$

Conversely, let $\omega \in R.H.S.$; then

$$\omega \in \bigcap_{i=1}^{n} E_i' \Longrightarrow \omega \notin E_i \text{ for any } i \Longrightarrow \omega \notin \bigcup_{i=1}^{n} E_i \Longrightarrow \omega \in \left(\bigcup_{i=1}^{n} E_i\right)'$$

(ii) We have from (i) that

$$\left(\bigcup_{i=1}^{n} E_i'\right)' = \bigcap_{i=1}^{n} \left(E_i'\right)' = \bigcap_{i=1}^{n} E_i \Longrightarrow \left(\bigcup_{i=1}^{n} E_i'\right) = \left(\bigcap_{i=1}^{n} E_i\right)'$$

Note: could also use induction on n.

[10 MARKS]

(b) Events E_1, E_2 and E_3 are general events in Ω.

(i) If E_1, E_2 and E_3 are pairwise (mutually) exclusive, then by axiom (III) we have

$$\mathrm{P}(E_1 \cup E_2 \cup E_3) = \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) = p_1 + p_2 + p_3$$

[2 MARKS]

(ii) Using the general addition formula, we have

$$\begin{aligned} \mathrm{P}(E_1 \cup E_2 \cup E_3) &= \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) \; - \; \mathrm{P}(E_1 \cap E_2) - \mathrm{P}(E_1 \cap E_3) - \mathrm{P}(E_2 \cap E_3) \; + \; \mathrm{P}(E_1 \cap E_2 \cap E_3) \\ &= p_1 + p_2 + p_3 - p_1p_2 - p_1p_3 - p_2p_3 + p_1p_2p_3 \end{aligned}$$

[4 MARKS]

The partition of Ω comprises $F_1, F_2, ..., F_8$, where

Event	Representation	Prob.
F_1	$E_1 \cap E_2 \cap E_3$	$p_1p_2p_3$
F_2	$E_1 \cap E_2 \cap E_3'$	$p_1p_2(1-p_3)$
F_3	$E_1 \cap E_2' \cap E_3$	$p_1(1-p_2)p_3$
F_4	$E_1 \cap E_2' \cap E_3'$	$p_1(1-p_2)(1-p_3)$
F_5	$E_1' \cap E_2 \cap E_3$	$(1-p_1)p_2p_3$
F_6	$E_1' \cap E_2 \cap E_3'$	$(1-p_1)p_2(1-p_3)$
F_7	$E_1' \cap E_2' \cap E_3$	$(1-p_1)(1-p_2)p_3$
F_8	$E_1' \cap E_2' \cap E_3'$	$(1-p_1)(1-p_2)(1-p_3)$

[4 MARKS]

EXERCISES

1.(a) $(S \cap T') \cup (S' \cap T)$ (b) $(D \cap (X \cup T) \cap S) \cup (((D \cap (X \cup T)') \cup D') \cap S')$

2. Let G="exactly one occurs". Then $G = (E \cap F') \cup (E' \cap F)$, and by Axiom (III) $\mathrm{P}(G) = \mathrm{P}(E \cap F') + \mathrm{P}(E' capF)$. Also,

$$\begin{aligned} E = (E \cap F) \cup (E \cap F') &\implies \mathrm{P}(E) = \mathrm{P}(E \cap F) + \mathrm{P}(E \cap F') \\ F = (E \cap F) \cup (E' \cap F) &\implies \mathrm{P}(F) = \mathrm{P}(E \cap F) + \mathrm{P}(E' \cap F) \end{aligned}$$

so therefore

$$\mathrm{P}(G) = \mathrm{P}(E) + \mathrm{P}(F) - 2\mathrm{P}(E \cap F).$$

3. (a) FALSE : (e.g. $E_1 = \emptyset$, $E_2 = \Omega$).

(b) FALSE : (e.g. coin toss, let $E_1 = E_2 = \{H\}$).

(c) TRUE : $E_1 \subseteq E_2 \implies \mathrm{P}(E_1) \leq \mathrm{P}(E_2) = 1 - \mathrm{P}(E_2^{'}) = 1 - \mathrm{P}(E_1) \implies 2\mathrm{P}(E_1) \leq 1$.

(d) TRUE : $\mathrm{P}(E_1 \cup E_2) = \mathrm{P}(E_1) + \mathrm{P}(E_1) - \mathrm{P}(E_1 \cap E_2) \geq (1 - x_1) + (1 - x_2) - 1 = 1 - x_1 - x_2$.

4. For general events E and F,
(a) $F \equiv (E \cap F) \cup (E' \cap F)$, so by Axiom (III)

$$\mathrm{P}(F) = \mathrm{P}(E \cap F) + \mathrm{P}(E' \cap F) \implies \mathrm{P}(E' \cap F) = \mathrm{P}(F) - \mathrm{P}(E \cap F)$$

(b) $E \cup F \equiv E \cup (E' \cap F)$, so by Axiom (III)

$$\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(E' \cap F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F)$$

(c) $E \subseteq F \implies F \equiv E \cup (E' \cap F)$, so by Axiom (III), as $\mathrm{P}(E' \cap F) \geq 0$,

$$\mathrm{P}(F) = \mathrm{P}(E) + \mathrm{P}(E' \cap F) \geq \mathrm{P}(E)$$

(d) *Bonferroni Inequality*: From (ii), as $\mathrm{P}(E \cup F) \leq 1$,

$$\mathrm{P}(E \cap F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cup F) \geq \mathrm{P}(E) + \mathrm{P}(F) - 1$$

5. (a) $E' \cup F' = (E \cap F)' \implies \mathrm{P}(E' \cup F') = 1 - \mathrm{P}(E_1 \cap E_2) = 1 - z$.

(b) $F = (E \cap F) \cup (E' \cap F)$, so $\mathrm{P}(F) = \mathrm{P}(E \cap F) + \mathrm{P}(E' \cap F)$, so $\mathrm{P}(E' \cap F) = y - z$.

(c) $E' \cup F = E' \cup (E \cap F) \implies \mathrm{P}(E' \cup F) = \mathrm{P}(E') + \mathrm{P}(E \cap F) = 1 - x + z$.

(d) $E' \cap F' = (E \cup F)' \implies \mathrm{P}(E' \cap F') = 1 - \mathrm{P}(E \cup F) = 1 - x - y + z$.

6. $E \cup E' = \Omega$, so by Axioms (II) and (III), $\mathrm{P}(E) + \mathrm{P}(E') = \mathrm{P}(\Omega) = 1$. Hence $\mathrm{P}(E') = 1 - \mathrm{P}(E)$. But, by Axiom (I)*, $\mathrm{P}(E') \geq 0$, hence $\mathrm{P}(E) \leq 1$.

7. (a) True for $n = 2$ as $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \leq \mathrm{P}(A) + \mathrm{P}(B)$. So assume true for $n = k$; then

$$\mathrm{P}(E_1 \cup ... \cup E_k \cup E_{k+1}) \leq \mathrm{P}(E_1 \cup ... \cup E_k) + \mathrm{P}(E_{k+1}) = \sum_{i=1}^{k+1} \mathrm{P}(E_i)$$

and hence true for $n = k + 1$.

(b) As for (a) except

$$\mathrm{P}(E_1 \cup ... \cup E_k \cup E_{k+1}) = \mathrm{P}(E_1 \cup ... \cup E_k) + \mathrm{P}(E_{k+1}) - \mathrm{P}\left((E_1 \cup ... \cup E_k) \cap E_{k+1}\right).$$

Result follows by substituting required form for $\mathrm{P}(E_1 \cup ... \cup E_k)$, and re-writing

$$\mathrm{P}\left((E_1 \cup ... \cup E_k) \cap E_{k+1}\right) = \mathrm{P}\left((E_1 \cap E_{k+1}) \cup ... \cup (E_k \cap E_{k+1})\right)$$

(using distributivity) which is the union of k events, and hence can use the inductive hypothesis to re-write this final expression in the required form.

TUTORIAL SHEET WEEK 3 : SOLUTIONS

1. Try to verify Axioms (I), (II) and (III) within each framework.

RELATIVE FREQUENCY: $\mathrm{P}(E) = \dfrac{n_E}{n}$ as $n \longrightarrow \infty$.

(I) OK, as n_E/n always lies in $[0, 1]$.

(II) OK, as $n_\Omega = n$, so $\mathrm{P}(\Omega) = n_\Omega/n = 1$.

(III) OK, as $n_{E\cup F} = (n_E + n_F)$ if E and F are disjoint, so $\mathrm{P}(E \cup F) = (n_E + n_F)/n = n_E/n + n_F/n = \mathrm{P}(E) + \mathrm{P}(F)$ as $n \longrightarrow \infty$.

CLASSICAL: Identical to RF case, but for finite n.

SUBJECTIVE:

(I) OK;

$$\begin{array}{ll} \mathrm{P}(E) > 1 & \Longrightarrow \text{ pay more than } M \text{ to win } M \text{ !} \\ \mathrm{P}(E) < 0 & \Longrightarrow \text{ paid to play !} \end{array}$$

(II) OK; if $E = \Omega$, must pay M so that neither side will be bound to lose.

(III) OK; Consider two bets on events E and F,

$$\left.\begin{array}{ll} \mathrm{P}(E) = p_E & \Longrightarrow \text{ pay } p_E M \text{ to win } M \text{ on } E \\ \mathrm{P}(F) = p_F & \Longrightarrow \text{ pay } p_F M \text{ to win } M \text{ on } F \end{array}\right\} \Longrightarrow \text{pay } (p_E + p_F)M \text{ to win } M \text{ on } E \text{ or } F$$

if E and F are mutually exclusive, so $\mathrm{P}(E \cup F) = (p_E + p_F) = \mathrm{P}(E) + \mathrm{P}(F)$.

2. Free choice, but suggest

(a) $\Omega = \{\mathrm{JAN}, \mathrm{FEB}, ..., \mathrm{DEC}\}$; equally likely outcomes ?

(b) $\Omega = \{x : 1889 \leq x \leq 2000\}$; relative frequency, subjective ?

(c) $\Omega = \{1001, 1002, ..., 5999, 6000\}$; equally likely outcomes.

(d) $\Omega = \{x : x \geq 0\}$; relative frequency ?

(e) $\Omega = \{x : -50 \leq x \leq 35\}$; relative frequency, subjective ?