M1S Probability and Statistics I

Spring Term 2000

The Course : The main results in the course are the ones relating to elementary aspects of Probability Theory; the Axioms, Conditional Probability and Independence, and the two Probability Theorems. Everything else that we covered was introduced to facilitate calculations based on these concepts. The Set Theory in Chapter 1 was introduced in order to build a framework in which to phrase and resolve probability questions. The Combinatorics techniques were introduced in Chapter 4 to allow calculations in "sample outcomes equally likely" problems, with generating functions giving a particularly straightforward method of solution. The material in Chapters 5 and 6 on discrete and continuous random variables and distributions gave a general and convenient way of expressing probability problems for experiments, and looked at probability functions (mass/density and distribution functions) in terms of their general properties, and the some specific examples of distributions relevant to standard experimental conditions. Chapter 7 introduced a general approach to transformations of random variables. Chapter 8 looked at expectations (including probability and moment generating functions) and their general properties, with the aim of finding another way of characterizing a probability distribution. Chapter 9 briefly studied joint probability distributions, with the main aim of looking at The Convolution Formula, and related methods for sums of random variables.

The Exam : Five questions in two hours. There will be a significant proportion of bookwork in each question, usually followed by an application of techniques or results from the bookwork part, of a similar type to the questions on the Exercise and Assessed work sheets.

For the exam, therefore, you will need KNOWLEDGE of the basic DEFINITIONS and THEOREMS contained in the course (to complete the bookwork bit), and an idea of how to APPLY the TECHNIQUES used in probability calculations. You will NOT have to remember, for example, forms of mass or density functions for specific distributions as these will be given where appropriate,

CHAPTER 1 : Sample Spaces and Events

1.1 Representing uncertainty in experimental contexts

1.2 Manipulating Collections of Sample Outcomes
- Sample outcomes
- Sample spaces (discrete/continuous, finite/countable/uncountable
- Events - occurrence, the Certain Event, the Impossible Event
- Set theory notation

1.3 Operations of Set Theory
- Binary Operations - Complement/Union/Intersection
- Exhaustive/Exclusive Events
- Elementary results - De Morgan's Laws
- Extensions of union/intersection ideas to more than two events - finite/countable unions
- Associative/Distributive Laws
- Representations of Complex Systems - component networks
- Disjoint unions/Partitions
- Construction of Partitions

CHAPTER 2 : Probability : Definitions, Interpretations, Basic Laws.

2.1 The Meaning of Probability
Interpretations of the Probability Function (relative frequency,classical,subjective)
Simple examples (0/1 experiments e.g. coin/drawing pin)
Specification of probabilities via Odds

2.2 The Mathematical Rules of Probability: The Probability Axioms
The Probability Axioms
Extensions to Axiom III to finite and countable collections of events

2.3 Corollaries to the Axioms
Probabilities for complement events
Probability of a union for non-exclusive events (The General Addition Rule)

CHAPTER 3 : Conditioning on New Information.

Conditional Probability - definition, interpretation
Examples of simple calculations
Independence for two events/ Mutual independence for more than two events
The General Multiplication (or Chain) Rule for Events (link to probability trees)

3.1 The Theorem of Total Probability
Statement and Proof of Theorem
Interpretation and Key elements
Application in simple and more complicated examples

3.2 Bayes Theorem
Statement and Proof of Theorem for two events/more than two events
Interpretation : Distinction between $P(A|B)$ and $P(B|A)$
Examples e.g. Medical Diagnosis, Simpson's Paradox
Prior and Posterior Odds
Conditional probability for more than two events
Conditional Independence

CHAPTER 4 : Combinatorics.

Enumeration for Equally Likely Outcomes : Combinatorics Problems (Sampling from a Finite Population, Occupancy Problems, Urn Models)

4.1 Counting Operations : Basic Methods and Terminology
Multiplication principle, Factorials
Distinguishable/Indistinguishable Objects; Sampling with/without replacement; Ordered/unordered outcomes.
Permutations/Combinations
Binary sequence representations

4.2 Combinatorial Identities

4.3 Partitioning : The Partition formula
Examples of Partitions (e.g. Poker hands)

4.4 Occupancy Problems (Distribution Problems)
Counting Techniques for Occupancy Problems involving
- Distinguishable objects - as a partitioning problem.
- Indistinguishable objects - using a binary sequence representation
Examples e.g. The Birthday Problem

4.5 Urn Models
The Hypergeometric Formula : two alternative justifications
Combinatorial/conditional probability justification
Uses for the Hypergeometric formula in probability calculations (for N, R, n or r varying)
Examples e.g. Fisher's Exact Test

4.6 Generating Functions for Combinatorics Calculations
Generating functions: definitions and uses
Solution of combinatorics problems using generating functions

CHAPTER 5 : Discrete Random Variables and Distributions

5.1 Random Variables
General definition
Discrete case (countable range)

5.2 Probability Mass Function
Definition, Notation, Interpretation
Properties
Examples

5.3 Discrete Cumulative Distribution Function
Definition, Notation, Interpretation
Properties
Connection with the probability mass function (summation/differencing)

5.4 Bernoulli Distribution

5.5 Binomial Distribution
Definition, experimental context, interpretation
Mass function Limiting behaviour as $n \to \infty, \theta \to 0$ with $n\theta = \lambda$ constant

5.6 Poisson Distribution
Definition, experimental context, interpretation
Mass function
Connection to the Poisson Process

5.7 Geometric Distribution
Mass function and CDF

5.8 Negative Binomial Distribution
Definition, experimental context, interpretation
Mass function
Connection to Geometric distribution

5.9 Hypergeometric Distribution
Definition, experimental context, interpretation
Mass function
Connection to Binomial distribution

5.10 Probability Generating Functions
Definition
Uses: calculations for sums of independent random variables.
Formulae for PGFs of standard distributions (eg. Geometric, Binomial, Poisson)

CHAPTER 6 : Continuous Random Variables and Distributions

6.1 Continuous Random Variables
General definition
Continuous probability specifications as the limit of discrete specifications

6.2 Continuous Cumulative Distribution Function
Definition, Notation, Interpretation
Properties

6.3 Probability Density Function
Definition, Notation, Interpretation
Properties
Connection with the continuous CDF function (integration/differentiation)

6.4 Continuous Uniform Distribution

6.5 Exponential Distribution
Definition, experimental context, interpretation
PDF and CDF
Connection to Poisson and Poisson Process
Connection to Uniform

6.6 Gamma Distribution
Definition, interpretation
The GAMMA function and properties
PDF
Special case: Chi-squared distribution $(\alpha = n/2, \beta = 1/2)\, \chi^2_n$
Connection to Exponential
Connection to Normal

6.7 Normal Distribution
Definition, experimental context, interpretation
PDF in standard and non–standard cases
Linear transformations

CHAPTER 7 : Transformations

Transformations of random variables $(Y = g(X))$
General transformation technique
Transformation Theorem for g 1-1

CHAPTER 8 : Expectations

Expectation of a discrete/continuous random variable
Expectations for general functions of random variables
Properties of Expectations
Linearity of Expectations
Probability and Moment Generating Functions as Expectations
Link between PGF and MGF
Calculations of PGF/MGF for standard distributions
Key results and uses of PGFs/MGFs

CHAPTER 9 : Joint Distributions

General joint discrete/continuous distributions
Discrete joint/marginal/conditional mass functions
Continuous joint/marginal/conditional density functions
Independence for random variables
Conditional Expectations
Bivariate Expectations
The Law of Iterated Expectations
Covariance/Correlation
The Convolution Formula
Expectations (and other calculation techniques) for Sums of Random Variables
The Central Limit Theorem