M1S : EXERCISES 4

COMBINATORICS AND PROBABILITY

1. The *binomial expansion* is used to express $(a+b)^n$ in power series form:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Use the binomial expansion to prove the following identities:

(a) For non-negative integer n,

$$1 - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} = 0$$

(b) For non-negative integers m and n, and for $0 \le k \le m+n$

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$$

2. The National Lottery each week randomly produces 6 winning numbers, drawn without replacement from the integers 1,2,...,49.

(a) If you have 1 entry ticket (i.e. one choice of 6 numbers), what is your chance of having the winning sequence of 6?

(b) A prize of £10 is won if exactly 3 of your numbers are in the winning sequence. What is the probability of such a win if you have 1 entry ticket?

3. If k random digits are chosen (with replacement) from $\{0, 1, 2, ..., 9\}$, what is the probability that

(a) all the digits chosen are less than or equal to r?

(b) the greatest digit chosen is equal to r ?

for each r, $0 \le r \le 9$.

4. Poker: In a hand of 5 cards dealt at random from a normal deck of 52 cards, what is the probability of obtaining:

(a) A Royal Flush (A, K, Q, J, 10 in the same suit)?

(b) Four of a Kind (e.g., 7, 7, 7, 7, K)?

(c) Two Pairs (e.g., A, A, 9, 9, J)?

5. A bridge hand consists of 13 cards from a normal deck of 52.

(a) How many different bridge hands are there?

(b) What is the probability that a randomly chosen bridge hand contains no aces, kings, queens or jacks?

6. n people, including yourself and a friend, are seated at random in a row of n chairs. What is the probability that you sit next to your friend?