M1S : EXERCISE SHEET 2

ELEMENTARY PROBABILITY

ASSESSED COURSEWORK
(Deadline : Tuesday 1st February)

(a) A more general version of the De Morgan laws given in lectures extends the result to more than two events. For a finite collection of events $E_1, E_2, ...E_n$ in sample space Ω,

$$\text{(i)} \quad \left(\bigcup_{i=1}^{n} E_i\right)' = \bigcap_{i=1}^{n} E_i' \qquad \text{(ii)} \quad \left(\bigcap_{i=1}^{n} E_i\right)' = \bigcup_{i=1}^{n} E_i'.$$

Prove (i) and (ii).

Hint: to show that two events A and B are identical you must show that

$$\omega \in A \iff \omega \in B$$

so for (i), first assume that

$$\omega \in \left(\bigcup_{i=1}^{n} E_i\right)' \tag{1}$$

and deduce that

$$\omega \in \bigcap_{i=1}^{n} E_i'. \tag{2}$$

Then assume (2) and deduce (1). For (ii), consider the collection of events $E_1', E_2', ..., E_n'$.

[*10 MARKS*]

(b) In (a), suppose that $n = 3$, and suppose that

$$\mathrm{P}(E_1) = p_1 \qquad \mathrm{P}(E_2) = p_2 \qquad \mathrm{P}(E_3) = p_3$$

Find an expression for $\mathrm{P}(E_1 \cup E_2 \cup E_3)$, assuming that

(i) E_1, E_2, E_3 are pairwise (or mutually) exclusive

[*2 MARKS*]

(ii) E_1, E_2, E_3 are **independent**, that is,

$$\mathrm{P}(E_i \cap E_j) = \mathrm{P}(E_i)\mathrm{P}(E_j) \;\; \text{for all } i \neq j$$

and

$$\mathrm{P}(E_1 \cap E_2 \cap E_3) = \mathrm{P}(E_1)\mathrm{P}(E_2)\mathrm{P}(E_3)$$

[*4 MARKS*]

Find a partition of Ω in terms of unions and intersections of E_1, E_2 and E_3, and write down the probabilities of the events in the partition assuming that E_1, E_2, E_3 are independent

[*4 MARKS*]

EXERCISES

1. A person selected from a population and subjected to screening-for-a particular-disease is either a sufferer (S) or not (S'). The screening will either result in a positive test (T), or a negative (T'). If the result of the test is used to classify the person, write down the event "person misclassified".

Suppose now that an X-ray is also taken and either gives a positive (X), or a negative (X') indication, and also a doctor carries out an examination resulting in a positive (D), or negative (D') assessment. If a person is classified as a sufferer if and only if the doctor and at least one of the tests points in this direction, represent the event "correct classification made".

2. Given two events $E, F \subseteq \Omega$, prove that the probability of *one and only one* of them occurring is

$$\mathrm{P}(E) + \mathrm{P}(F) - 2\mathrm{P}(E \cap F).$$

3. Consider the following statements, which are claimed to be true for events E_1, E_2 in a sample space Ω:

$$\begin{array}{lll}
(a) & \mathrm{P}(E_1) = 0 & \Longrightarrow \mathrm{P}(E_1 \cup E_2) = 0 \\
(b) & \mathrm{P}(E_1) = \mathrm{P}(E_2') & \Longrightarrow E_1' = E_2 \\
(c) & E_1 \subseteq E_2 \text{ and } \mathrm{P}(E_1) = P(E_2') & \Longrightarrow P(E_1) \leq 1/2 \\
(d) & \mathrm{P}(E_1') = x_1, P(E_2') = x_2 & \Longrightarrow P(E_1 \cup E_2) \geq 1 - x_1 - x_2
\end{array}$$

In each case, either prove that the statement is true for all Ω, E_1, E_2, or provide a specific counter-example to show that there exists Ω, E_1, E_2 for which it is false

4. Given two events $E, F \subseteq \Omega$, prove that

$$\begin{array}{llll}
(a) & \mathrm{P}(E' \cap F) = \mathrm{P}(F) - \mathrm{P}(E \cap F) & (b) & \mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F) \\
(c) & E \subseteq F \Longrightarrow \mathrm{P}(E) \leq \mathrm{P}(F) & (d) & \mathrm{P}(E \cap F) \geq \mathrm{P}(E) + \mathrm{P}(F) - 1
\end{array}$$

[(d) is known as *Bonferroni's Inequality.*]

5. Suppose that E and F are events such that $\mathrm{P}(E) = x, \mathrm{P}(F) = y$ and $\mathrm{P}(E \cap F) = z$. Express the following terms of x, y and z:

$$\begin{array}{llll}
(a) & P(E' \cup F') & (b) & P(E' \cap F) \\
(c) & P(E' \cup F) & (d) & P(E' \cap F').
\end{array}$$

6. Show that the first probability axiom can be relaxed to:

$$(\mathrm{I})^* : \mathrm{P}(E) \geq 0 \text{ for all } E \subseteq \Omega,$$

that is, given that $(\mathrm{I})^*$, (II) and (III) hold, prove that $\mathrm{P}(E) \leq 1$ for all $E \subseteq \Omega$.

7. Use induction to prove that, for any events $E_1, E_2, ..., E_n$ in Ω:

(a) $\mathrm{P}(E_1 \cup E_2 \cup ... \cup E_n) \leq P(E_1) + ... + P(E_n)$ $(Boole's\ Inequality)$

(b) $$\mathrm{P}(E_1 \cup E_2 \cup ... \cup E_n) = \sum_i \mathrm{P}(E_i) - \sum_i \sum_j \mathrm{P}(E_i \cap E_j) + \sum_i \sum_j \sum_k \mathrm{P}(E_i \cap E_j \cap E_k) - ... + (-1)^{n-1} \mathrm{P}(E_1 \cap E_2 \cap ... \cap E_n)$$

In (b), summations are over *distinct* sets of subscripts, each counted once only. For example, if $n = 3$, the result would correspond to

$$\mathrm{P}(E_1 \cup E_2 \cup E_3) = \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) - \mathrm{P}(E_1 \cap E_2) - \mathrm{P}(E_1 \cap E_3) - \mathrm{P}(E_2 \cap E_3) + \mathrm{P}(E_1 \cap E_2 \cap E_3).$$