M1S : EXERCISES 10

JOINT DISTRIBUTIONS AND SUMS OF RANDOM VARIABLES

1. Two fair dice are rolled. let X and Y be the discrete random variables corresponding to the largest and smallest of the two scores respectively. Find the joint mass function of X and Y, $f_{X,Y}$, the marginal mass functions of X and Y, f_X and f_Y , and the conditional mass function of Y given that $X = x$ for $x \in \{1, 2, 3, 4, 5, 6\}$.

Are X and Y independent ? Justify your answer.

2. Two cards are drawn without replacement from a standard pack of 52. Let X and Y be the discrete random variables corresponding to the numbers of kings and the number of queens drawn respectively. Find the joint mass function of X and Y,

$$f_{X,Y}(x, y) = \mathrm{P}[\ X = x, Y = y\ ]$$

for x and y taking values $0, 1, 2$.

Are X and Y independent ? Justify your answer.

3. Continuous random variables X and Y have joint probability density function $f_{X,Y}$ specified by

$$f_{X,Y}(x, y) = y^{-1} \exp\left\{-y - xy^{-1}\right\} \qquad x, y > 0$$

and zero otherwise.

Show that Y has an exponential distribution.

4. Continuous random variables X and Y have joint probability density function $f_{X,Y}$ specified by

$$f_{X,Y}(x, y) = x^{-1} \qquad 0 \leq y \leq x \leq 1$$

and zero otherwise.

Show that, marginally, $X \sim Uniform(0, 1)$, and that conditional on $X = x$ for $0 \leq x \leq 1$, $Y \sim Uniform(0, x)$.

Calculate $\mathrm{P}[\ Y \leq y \mid X = x\ ]$ for $0 \leq y \leq x$.

5. Continuous random variables X and Y have joint probability density function $f_{X,Y}$ specified by

$$f_{X,Y}(x, y) = c(x + 2y) \qquad 0 \leq x \leq 1, 0 \leq y \leq 1$$

and zero otherwise, for some constant c. Find

(i) the value of constant c.

(ii) $\mathrm{P}[X \leq 1/2, Y \leq 1/3]$.

(iii) the marginal densities of X and Y.

(iv) $\mathrm{P}[\ Y \leq 1/3 \mid X \leq 1/2\ ]$.

6. Continuous random variables $X_1 \sim Exponential(\lambda_1)$ and $X_2 \sim Exponential(\lambda_2)$ are independent. Find the distribution of

$$\text{(i) } U = \text{Min}\{X_1, X_2\} \qquad \text{(ii) } V = \text{Max}\{X_1, X_2\} \qquad \text{(iii) } T = X_1 + X_2.$$

[*Hint: for (i),* $F_U(u) = P[U \leq u]$ *and*

$$\begin{aligned} P[U \leq u] &= P[Min\{X_1, X_2\} \leq u] = 1 - P[Min\{X_1, X_2\} > u] = 1 - P[X_1 > u, X_2 > u] \\ &= 1 - P[X_1 > u]P[X_2 > u] = 1 - (1 - F_{X_1}(u))(1 - F_{X_2}(u)) \end{aligned}$$

as X_1 *and* X_2 *are independent, where* F_{X_1} *and* F_{X_2} *are the cumulative distribution functions of* X_1 *and* X_2 *respectively. An amended argument can be used for (ii).*]

7. *The Convolution Theorem*

If X and Y have a joint distribution specified via $f_{X,Y}$ it is possible to compute the probability distribution of $T = X + Y$ using the convolution theorem. If X and Y are discrete, the mass function of T is given by

$$f_T(t) = \sum_{x=-\infty}^{\infty} f_{X,Y}(x, t - x) \qquad \text{or} \qquad f_T(t) = \int_{-\infty}^{\infty} f_{X,Y}(x, t - x)dx$$

in the discrete and continuous cases respectively, and where simplifications occur if X and Y are independent.

Use the convolution theorem to show the following

$$\begin{array}{lll} X \sim Binomial(n_X, \theta) & Y \sim Binomial(n_Y, \theta) & \Longrightarrow T \sim Binomial(n_X + n_Y, \theta) \\ X \sim Poisson(\lambda_1) & Y \sim Poisson(\lambda_2) & \Longrightarrow T \sim Poisson(\lambda_1 + \lambda_2) \\ X \sim Geometric(\theta) & Y \sim Geometric(\theta) & \Longrightarrow T \sim NegBinomial(2, \theta) \\ X \sim Exponential(\lambda) & Y \sim Exponential(\lambda) & \Longrightarrow T \sim Gamma(2, \lambda) \\ X \sim Normal(\mu_X, \sigma_X^2) & Y \sim Normal(\mu_Y, \sigma_Y^2) & \Longrightarrow T \sim Normal(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2) \end{array}$$

8. *The Central Limit Theorem.*

(i) Using the central limit theorem, evaluate the approximate probability that, in 20 independent rolls of a fair dice, the sum of the 20 scores lies between 30 and 40.

(ii) Show that the Binomial, Poisson, Negative Binomial, and Gamma distributions can, in certain circumstances, be well approximated by appropriately chosen Normal distributions.