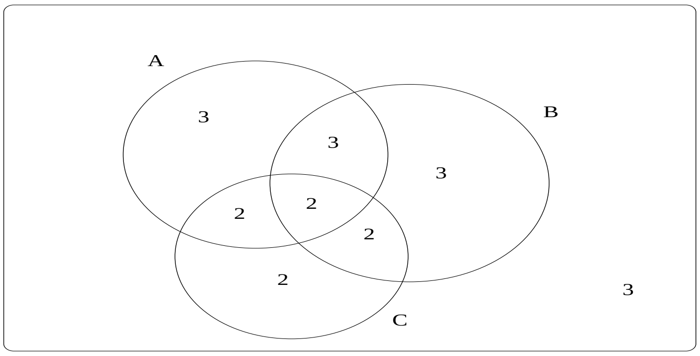
CHEM. ENG. II - PROBABILITY AND STATISTICS

SOLUTIONS

EXERCISES 1

1. Events A="even numbered", B="blue ball", C="number less than 5".



(1) $A \cap B \cap C$ - "blue ball 2 or 4" : Prob = $2/20$.
(2) $B \cap C'$ - "blue ball 5,6,7,8,9,19" : Prob = $6/20$.
(3) $A \cup B \cup C$ - "at least one of blue, or even, or less than 5" : Prob = $17/20$.
(4) $A \cap (B \cup C)$ - "even, and either blue or ess than 5" : Prob = $7/20$.
(5) $A' \cap B' \cap C'$ - "red 5,7,9" - Prob = $3/20$.

2. Table 1:

$$\left.\begin{aligned}
\mathrm{P}(E) &= \mathrm{P}(E \cap F) + \mathrm{P}(E \cap F') &&= \frac{6}{24} + \frac{2}{24} = \frac{8}{24} = \frac{1}{3} \\
\mathrm{P}(F) &= \mathrm{P}(E \cap F) + \mathrm{P}(E' \cap F) &&= \frac{6}{24} + \frac{12}{24} = \frac{18}{24} = \frac{3}{4} \\
\mathrm{P}(F|E) &= \frac{\mathrm{P}(E \cap F)}{\mathrm{P}(F)} &&= \frac{6/24}{8/24} = \frac{3}{4} = \mathrm{P}(F) \\
\mathrm{P}(F|E') &= \frac{\mathrm{P}(E' \cap F)}{\mathrm{P}(E')} &&= \frac{12/24}{16/24} = \frac{3}{4} = \mathrm{P}(F)
\end{aligned}\right\} \Longrightarrow \text{INDEPENDENT}$$

Table 2:

$$\left.\begin{aligned}
\mathrm{P}(E) &= \mathrm{P}(E \cap F) + \mathrm{P}(E \cap F') &&= \frac{6}{28} + \frac{4}{28} = \frac{10}{28} = \frac{5}{14} \\
\mathrm{P}(F) &= \mathrm{P}(E \cap F) + \mathrm{P}(E' \cap F) &&= \frac{6}{28} + \frac{12}{28} = \frac{18}{28} = \frac{9}{14} \\
\mathrm{P}(F|E) &= \frac{\mathrm{P}(E \cap F)}{\mathrm{P}(F)} &&= \frac{6/28}{10/28} = \frac{3}{5} \neq \mathrm{P}(F) \\
\mathrm{P}(F|E') &= \frac{\mathrm{P}(E' \cap F)}{\mathrm{P}(E')} &&= \frac{12/28}{18/28} = \frac{2}{3} \neq \mathrm{P}(F)
\end{aligned}\right\} \Longrightarrow \begin{array}{c}\text{NOT}\\ \text{INDEPENDENT}\end{array}$$

Check also that $\mathrm{P}(E \cap F) = \mathrm{P}(E)\mathrm{P}(F|E)$ etc. for both tables.

3. $\mathrm{P}(F) = \mathrm{P}(A \cap B \cap C \cap D \cap E) = \mathrm{P}(A)\mathrm{P}(B|A)\mathrm{P}(C|A \cap B)\mathrm{P}(D|A \cap B \cap C)\mathrm{P}(E|A \cap B \cap C \cap D)$

(a) A, B, C, D, E independent, $\mathrm{P}(A) = \mathrm{P}(B) = \mathrm{P}(C) = \mathrm{P}(D) = \mathrm{P}(E) = 0.90$. so

$$\mathrm{P}(F) = (0.9)^5 = 0.5905$$

(b) Theorem of Total Probability:

$$\mathrm{P}(B) = \mathrm{P}(B|A)\mathrm{P}(A) + \mathrm{P}(B|A')\mathrm{P}(A') = (0.95 \times 0.9) + (0.75 \times 0.1) = 0.93.$$

(c) $\mathrm{P}(F) = \mathrm{P}((A \cup B) \cap C \cap D \cap E) = \mathrm{P}(A \cup B)\mathrm{P}(C)\mathrm{P}(D)\mathrm{P}(E)$ where

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

(i) A, B independent, so $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) \implies \mathrm{P}(A \cup B) = 0.9 + 0.9 - 0.9^2 = 0.99$,

$$\implies \mathrm{P}(F) = 0.99 \times (0.9)^3 = 0.7217$$

(ii) A, B not independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A) \implies \mathrm{P}(A \cup B) = 0.9 + 0.93 - 0.90 \times 0.95 = 0.975$,

$$\implies \mathrm{P}(F) = 0.975 \times (0.9)^3 = 0.7108$$

4. Given that $\mathrm{P}(S) = 0.001, \mathrm{P}(T|S) = 1.0, \mathrm{P}(T|S') = 0.1$.

(i) Theorem of Total Probability:

$$\mathrm{P}(T) = \mathrm{P}(T|S)\mathrm{P}(S) + \mathrm{P}(T|S')\mathrm{P}(S') = (1.0 \times 0.001) + (0.01 \times 0.999) = 0.1009.$$

(ii) Let $I =$ "incorrect diagnosis"; then $I = (T \cap S') \cup (T' \cap S)$,

$$\implies \mathrm{P}(I) = \mathrm{P}(T \cap S') + \mathrm{P}(T' \cap S) = \mathrm{P}(T|S')\mathrm{P}(S') + \mathrm{P}(T|S)\mathrm{P}(S) = (0.1 \times 0.999) + (0.0 \times 0.001) = 0.0999.$$

(iii) Bayes Theorem:

$$\mathrm{P}(S|T) = \frac{\mathrm{P}(T|S)\mathrm{P}(S)}{\mathrm{P}(T)} = \frac{1.0 \times 0.001}{0.1009} = 0.0099$$

5. First let $E =$ "sequence of n heads", $F=$ "coin is normal". Want

$$\mathrm{P}(F|E) = \frac{\mathrm{P}(E|F)\mathrm{P}(F)}{\mathrm{P}(E)} = \frac{\mathrm{P}(E|F)\mathrm{P}(F)}{\mathrm{P}(E|F)\mathrm{P}(F) + \mathrm{P}(E|F')\mathrm{P}(F')}$$

and here $\mathrm{P}(F) = 1/2$, $\mathrm{P}(E|F) = (1/2)^n$, $\mathrm{P}(E|F') = 1$, so

$$\mathrm{P}(F|E) = \frac{1}{1 + 2^n}$$

Now let $E =$ "sequence of k heads and $n - k$ tails. Now

$$\mathrm{P}(E|F) = \binom{n}{k}(1/2)^n \qquad \mathrm{P}(E|F') = \binom{n}{k}p^k(1-p)^{n-k}$$

so

$$\mathrm{P}(E) = \mathrm{P}(E|F)\mathrm{P}(F) + \mathrm{P}(E|F')\mathrm{P}(F') = \binom{n}{k}(1/2)^n \times 1/2 + \binom{n}{k}p^k(1-p)^{n-k} \times 1/2$$

and

$$\mathrm{P}(F|E) = \frac{1}{1 + 2^n p^k (1-p)^{n-k}}$$

6. Let E="flower produces ripe fruit, F="flower pollinated and fruit produced", G="fruit ripens". Then, assuming independence,

$$\mathrm{P}(E) = \mathrm{P}(F \cap G) = \mathrm{P}(F)\mathrm{P}(G) = 2/3 \times 3/4 = 1/2$$

Now let A_n be the event that the tree produces n flowers, and B_r be the event that the tree produces r ripe fruit (for $n \geq r$). Then

$$\mathrm{P}(A_n|B_r) = \frac{\mathrm{P}(B_r|A_n)\mathrm{P}(A_n)}{\sum\limits_{n=r}^{\infty}\mathrm{P}(B_r|A_n)\mathrm{P}(A_n)}$$

Now

$$\mathrm{P}(B_r|A_n) = \binom{n}{r}\left(\frac{1}{2}\right)^n \qquad \mathrm{P}(A_n) = (1-p)p^n$$

so

$$\mathrm{P}(A_n|B_r) = \frac{\binom{n}{r}\left(\frac{1}{2}\right)^n (1-p)p^n}{\sum\limits_{n=r}^{\infty}\binom{n}{r}\left(\frac{1}{2}\right)^n (1-p)p^n} = \binom{n}{r}\left(\frac{p}{2}\right)^n \frac{1}{\sum\limits_{x=0}^{\infty}\binom{x+r}{r}\left(\frac{p}{2}\right)^{x+r}} = \binom{n}{r}\left(\frac{p}{2}\right)^{n-r}\frac{1}{\left(1-\frac{p}{2}\right)^{-(r+1)}}$$

using the binomial expansion for negative exponent, that is, for $r \geq 0$ and $-1 < t < 1$,

$$(1-t)^{-(r+1)} = \frac{1}{(1-t)^{r+1}} = \sum_{x=0}^{\infty}\binom{x+r}{r}t^{x+r}$$

Hence

$$\mathrm{P}(A_n|B_r) = \binom{n}{r}\frac{p^{n-r}(2-p)^{r+1}}{2^{n+1}} \qquad r \leq n$$

7. Let A, B and C represent the events that the prize lies behind the selected door, the opened door, and the remaining door respectively, and let H_B represent the event that the host opens door the second door. Want to compare the conditional probabilities of A and C given H_B. Now,

$$\mathrm{P}(A|H_B) = \frac{\mathrm{P}(H_B|A)\mathrm{P}(A)}{\mathrm{P}(H_B)} = \frac{\mathrm{P}(H_B|A)\mathrm{P}(A)}{\mathrm{P}(H_B|A)\mathrm{P}(A)+\mathrm{P}(H_B|B)\mathrm{P}(B)+\mathrm{P}(H_B|C)\mathrm{P}(C)}.$$

Here $\mathrm{P}(A) = \mathrm{P}(B) = \mathrm{P}(C) = 1/3$, and $\mathrm{P}(H_B|A) = 1/2$ (host can choose to open B or C), $\mathrm{P}(H_B|B) = 0$ (host cannot open B if the prize is behind it) and $\mathrm{P}(H_B|C) = 1$ (host must open B, as A is already selected). Hence

$$\mathrm{P}(A|H_B) = \frac{1/2 \times 1/3}{1/2 \times 1/3 + 0 \times 1/3 + 1 \times 1/3} = 1/3.$$

Similarly, $\mathrm{P}(C|H_B) = 2/3$, so it is **always** probabilistically advantageous to SWITCH doors as $\mathrm{P}(A|H_B) < \mathrm{P}(C|H_B)$.

EXERCISES 2

1. (a) X="operating (0), not operating (1)": $\mathbb{X} = \{0, 1\}$
   (b)X="number of defective components": $\mathbb{X} = \{0, 1, 2, ..., 100\}$.
   (c)X="number of failed components": $\mathbb{X} = \{0, 1, 2, ...\}$.
   (d)X="time at which 100th component fails": $\mathbb{X} = \{x : x \geq 0\}$.

2. (a) OK
   (b) Not a probability distribution ($f_X(2) < 0$)
   (c) OK
   (d) Not a probability distribution (doesn't sum to 1).

3. (a) OK
   (b) OK
   (c) Not a probability distribution (integral is not finite)
   (d) OK

4. $f_X(0) = F_X(0) = 1/2$, and for $x = 1, 2, 3, ...$

$$f_X(x) = F_X(x) - F_X(x-1) = (1 - (1/2)^{x+1}) - (1 - (1/2)^x) = (1/2)^{x+1}$$

Finally

$$\mathrm{P}[X \text{ is even }] = \mathrm{P}[X = 0] + \mathrm{P}[X = 2] + \mathrm{P}[X = 4] + ... = (1/2) + (1/2)^3 + (1/2)^5 + ...$$

as $f_X(x) = \mathrm{P}[X = x]$. This is an infinite geometric progression with common term $(1/2)^2$. Hence

$$\mathrm{P}[X \text{ is even }] = 1/2 \times \frac{1}{1 - (1/2)^2} = 2/3$$

5. Need $\displaystyle\int_0^1 f_X(x)\ dx = 1 \Longrightarrow \int_0^1 c(1-x)x^2 dx = 1 \Longrightarrow c\left[\ x^3/3 - x^4/4\right]_0^1 = 1 \Longrightarrow c = 12.$

CDF obtained by integrating the density function:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_0^x 12(1-t)t^2 dt = 4x^3 - 3x^4 \qquad 0 \leq x \leq 1$$

and hence $\mathrm{P}[1/4 < X < 3/4] = F_X(3/4) - F_X(1/4) = \dfrac{176}{256}$

6. (a) By integration, $c = 1/(\alpha - \beta)$, that is, need

$$\int_0^1 f_X(x)\ dx = 1$$

   (b) By differentiation

$$f_X(x) = \alpha \sec^2 x \exp\{-\alpha \tan x\} \qquad 0 \leq x \leq 2\pi$$

   (c) By differentiation

$$f_X(x) = \frac{2\beta(\beta^2 - x^2)}{(\beta^2 + x^2)^2} \qquad 0 \leq x \leq \beta$$

7. X_1=sum of scores; takes values on $\{2,3,...,12\}$. X_2=largest of the two scores; takes values on $\{1,2,3,4,5,6\}$. Joint distribution specified by

		X_1											
		2	3	4	5	6	7	8	9	10	11	12	Tot
	1	p_1	0	0	0	0	0	0	0	0	0	0	p_1
	2	0	p_2	p_1	0	0	0	0	0	0	0	0	p_1+p_2
	3	0	0	p_2	p_2	p_1	0	0	0	0	0	0	p_1+2p_2
X_2	4	0	0	0	p_2	p_2	p_2	p_1	0	0	0	0	p_1+3p_2
	5	0	0	0	0	p_2	p_2	p_2	p_2	p_1	0	0	p_1+4p_2
	6	0	0	0	0	0	p_2	p_2	p_2	p_2	p_2	p_1	p_1+5p_2
Tot		p_1	p_2	p_1+p_2	$2p_2$	p_1+2p_2	$3p_2$	p_1+2p_2	$2p_2$	p_1+p_2	p_2	p_1	1

where $p_1 = 1/36$, $p_2 = 2/36$. Marginal for X_1 given by column totals, so that

x	2	3	4	5	6	7	8	9	10	11	12
$f_{X_1}(x)$	p_1	p_2	p_1+p_2	$2p_2$	p_1+2p_2	$3p_2$	p_1+2p_2	$2p_2$	p_1+p_2	p_2	p_1

Marginal for X_2 given by row totals, so that

x	1	2	3	4	5	6
$f_{X_2}(x)$	p_1	p_1+p_2	p_1+2p_2	p_1+3p_2	p_1+4p_2	p_1+5p_2

Conditional for X_1, given $X_2 = 4$ given by the fourth row of the table, divided by the fourth row total.

8. (a) Marginal for X_1 given by

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 = \int_0^1 4x_1x_2\ dx_2 = 2x_1 \int_0^1 2x_2\ dx_2 = 2x_1 \qquad 0 < x_1 < 1$$

and by symmetry,

$$f_{X_2}(x_2) = 2x_2 \qquad 0 < x_2 < 1$$

(b) Marginal for X_1 given by

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\ dx_2 = \int_0^1 (x_1+x_2)\ dx_2 = \left[x_1x_2 + x_2^2/2\right]_0^1 = x_1 + \frac{1}{2} \qquad 0 < x_1 < 1$$

and by symmetry

$$f_{X_2}(x_2) = x_2 + \tfrac{1}{2} \qquad 0 < x_2 < 1$$

EXERCISES 3

1. $X \sim Bin(10, 0.2)$, so $\mathbb{X} = \{0, 1, 2, ..., 10\}$, $f_X(x) = \mathrm{P}[X = x] = \binom{10}{x}(0.2)^x(0.8)^{10-x}$

x	0	1	2	3	4	5	6	7	8	9	10
$f_X(x)$	0.107	0.268	0.302	0.201	0.088	0.026	0.001	0.000	0.000	0.000	0.000

2. X="number of heads", so $X \sim Bin(5, 0.5)$
(i) $\mathrm{P}[X = 0] = (0.5)^5 = 0.03125$
(ii) $\mathrm{P}[X \geq 4] = \mathrm{P}[X = 4] + \mathrm{P}[X = 5] = \binom{5}{4}(0.5)^5 + \binom{5}{5}(0.5)^5 = 0.1875$
(iii) $\mathrm{P}[\text{Odd number of heads}] = \mathrm{P}[X = 1] + \mathrm{P}[X = 3] + \mathrm{P}[X = 5] = 0.5$

3. X="number of defectives", so $X \sim Bin(50, 0.01)$
(i) $\mathrm{P}[X \leq 2] = \mathrm{P}[X = 0] + \mathrm{P}[X = 1] + \mathrm{P}[X = 2]$, so

$$\mathrm{P}[X \leq 2] = \binom{50}{0}(0.01)^0(0.99)^{50} + \binom{50}{1}(0.01)^1(0.99)^{49} + \binom{50}{2}(0.01)^2(0.99)^{48} = 0.9862$$

(ii) If X_1 is the Bernoulli random variable corresponding to component 1, want $\mathrm{P}[X_1 = 1|X = 3]$.

$$\mathrm{P}[X_1 = 1|X = 3] = \frac{\mathrm{P}[X = 3|X_1 = 1]\mathrm{P}[X_1 = 1]}{\mathrm{P}[X = 3]} = \frac{\mathrm{P}[Y = 2]\mathrm{P}[X_1 = 1]}{\mathrm{P}[X = 3]}$$

where Y is the random variable corresponding to the total number of defectives in components 2,...,49. Now $Y \sim Bin(49, 0.01)$, so

$$\mathrm{P}[X_1 = 1|X = 3] = \frac{\binom{49}{2}(0.01)^2(0.99)^{47} \times 0.01}{\binom{50}{3}(0.01)^3(0.99)^{47}} = \frac{3}{50}$$

4. $X \sim Poisson(\lambda) \Longrightarrow p_0 = e^{-\lambda},\ p_x = \dfrac{\lambda}{x}p_{x-1}$

x	0	1	2	3	4	5	6	7	8	9	10
$f_X(x)$	0.007	0.034	0.084	0.140	0.175	0.175	0.146	0.104	0.065	0.036	0.018

5. Must check pdf is non-negative, and $\int_0^\infty f_X(x)\ dx = 1$; density decays exponentially from maximum value λ at $x = 0$.
Can show $F_X(x) = \mathrm{P}[X \leq x] = 1 - e^{-\lambda x}$ for $x \geq 0$, so

$$\mathrm{P}[X \leq 1] = 1 - e^{-2} = 0.8647 \qquad \mathrm{P}[X > 4] = 1 - \mathrm{P}[X \leq 4] = e^{-8} = 3.35 \times 10^{-4}$$

6. First consider $Y_2 = X_1 + X_2$; by the convolution theorem

$$f_{Y_2}(y) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y - x_1)\ dx_1 = \int_0^y \lambda e^{-\lambda x_1} \lambda e^{-\lambda(y-x_1)}\ dx_1 = \lambda^2 e^{-\lambda y} \int_0^y\ dx_1 = \lambda^2 y e^{-\lambda y}$$

Assume true for $n = k-1$, and then re-write $Y_k = Y_{k-1} + X_k$ and use the convolution theorem

$$\begin{aligned} f_{Y_k}(y) &= \int_{-\infty}^{\infty} f_{Y_{k-1}}(x_1) f_{X_k}(y - x_1)\ dx_1 = \int_0^y \frac{\lambda^{k-1}}{(k-2)!} x_1^{k-2} e^{-\lambda x_1} \lambda e^{-\lambda(y-x_1)}\ dx_1 \\ &= \frac{\lambda^k}{(k-2)!} e^{-\lambda y} \int_0^y x_1^{k-2}\ dx_1 = \frac{\lambda^k}{(k-1)!} y^{k-1} e^{-\lambda y} \qquad y > 0. \end{aligned}$$

and hence the result follows by induction.

6. To compute expectation and variance, need to integrate

$$\mathrm{E}_{f_X}[X] = \int_{-\infty}^{\infty} x f_X(X)\ dx \qquad \mathrm{E}_{f_X}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(X)\ dx.$$

This is achieved by changing variables in the integrals from x to $t = (x - \mu)/\sigma$. The result is finally obtained by noting that

$$\int_{-\infty}^{\infty} e^{-t^2/2}\ dt = \sqrt{2\pi} \qquad \text{(as the integrand is proportional to the standard normal pdf)}$$

$$\int_{-\infty}^{\infty} t e^{-t^2/2}\ dt = 0 \qquad \text{(as the integrand is an ODD function)}$$

$$\int_{-\infty}^{\infty} t^2 e^{-t^2/2}\ dt = \sqrt{2\pi} \quad \text{(using a further substitution } u = t^2\text{)}$$

Hence $\mathrm{E}_{f_X}[X] = \mu \qquad \mathrm{E}_{f_X}[X^2] = \mu^2 + \sigma^2 \Longrightarrow \mathrm{Var}_{f_X}[X] = \sigma^2$.

7. $F_Y(y) = \mathrm{P}[Y \le y] = \mathrm{P}[\sigma X + \mu \le y] = \mathrm{P}[X \le (y-\mu)/\sigma] = \Phi\left(\dfrac{y-\mu}{\sigma}\right)$, and so by differentiation

$$f_Y(y) = \frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}$$

where Φ and ϕ are the cdf and pdf of the standard normal distribution.

8. If $X \sim N(\mu, \sigma^2)$, $F_X(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$. Note also that

$$\mathrm{P}[X > x] = 1 - \mathrm{P}[X \le x] \qquad \mathrm{P}[x_1 < X \le x_2] = \mathrm{P}[X \le x_2] - \mathrm{P}[X \le x_1]$$

and for $x < 0$,

$$\Phi(x) = 1 - \Phi(-x).$$

Then

	$\mu = 0, \sigma^2 = 1$	$\mu = 0.5, \sigma^2 = 2.$
(i) $\mathrm{P}[X \leq 2.2]$	0.9861	0.8023
(ii) $\mathrm{P}[X > -0.75]$	0.7734	0.6368
(iii) $\mathrm{P}[-1.7 \leq X < -1]$	0.1141	0.0909
(iv) $\mathrm{P}[X > 1.96]$	0.0250	0.2327

9. $Y \sim \chi^2_1 \Longleftrightarrow Y = X^2$ where $X \sim N(0,1)$. Hence

$$\mathrm{P}[\ Y > y\ ] = \mathrm{P}[\ X^2 > y\ ] = 1 - \mathrm{P}[\ -\sqrt{y} < X < \sqrt{y}\ ] = 2(1 - \Phi(\sqrt{y}))$$

10. Use Central Limit Theorem to give approximate probabilities in Binomial, Poisson, Gamma cases, that is,

$$X \sim Bin(n,\theta) \qquad \Longrightarrow \quad X \sim N(n\theta, n\theta(1-\theta)) \text{ approximately, } F_X(x) \approx \Phi\left(\frac{x - n\theta}{\sqrt{\theta(1-\theta)}}\right).$$

$$X \sim Poisson(\lambda) \qquad \Longrightarrow \quad X \sim N(\lambda, \lambda) \text{ approximately, } F_X(x) \approx \Phi\left(\frac{x-\lambda}{\sqrt{\lambda}}\right).$$

$$X \sim Gamma(n,\lambda) \qquad \Longrightarrow \quad X \sim N(n/\lambda, n/\lambda^2) \text{ approximately, } F_X(x) \approx \Phi\left(\frac{x - n/\lambda}{\sqrt{n/\lambda^2}}\right).$$

Hence, from Normal tables (ignoring any continuity correction)

(i) $\mathrm{P}[X \leq 200] \approx \Phi\left(\frac{200 - 1000 \times 0.2}{\sqrt{1000 \times 0.2 \times 0.8}}\right) = \Phi(0) = 0.500$

(ii) $\mathrm{P}[X > 2275] = 1 - \mathrm{P}[X \leq 2275] \approx 1 - \Phi\left(\frac{2275 - 2500 \times 0.9}{\sqrt{2500 \times 0.9 \times 0.1}}\right) = \Phi(1.67) = 1 - 0.9525 = 0.048.$

(iii) $\mathrm{P}[725 \leq X \leq 740] \approx \Phi\left(\frac{740 - 1500 \times 0.5}{\sqrt{1500 \times 0.5 \times 0.5}}\right) - \Phi\left(\frac{725 - 1500 \times 0.5}{\sqrt{1500 \times 0.5 \times 0.5}}\right)$

$= \Phi(-0.52) - \Phi(-1.29) = (1 - \Phi(0.52)) - (1 - \Phi(1.29)) \approx (1 - 0.6985) - (1 - 0.9015) = 0.203.$

(iv) $\mathrm{P}[X \leq 190] \approx \Phi\left(\frac{190 - 200}{\sqrt{200}}\right) = \Phi(-0.71) = (1 - \Phi(0.71) = 1 - 0.7611 = 0.239$

(v) $\mathrm{P}[X > 120] \approx 1 - \mathrm{P}[X \leq 120] \approx 1 - \Phi\left(\frac{120 - 100}{\sqrt{100}}\right) = 1 - \Phi(2.00) = 1 - 0.9772 \approx 0.023.$

(vi) $\mathrm{P}[X \leq 12] \approx \Phi\left(\frac{12 - 20/2}{\sqrt{20/2^2}}\right) = \Phi(0.89) = 0.813$

EXERCISES 4

1. Maximum likelihood for λ;

STEP 1 Likelihood function $L(\lambda)$, for $\lambda > 0$,

$$L(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n \exp\left\{-\lambda \sum_{i=1}^{n} x_i\right\}$$

STEP 2 Log-likelihood $\log L(\lambda)$.

$$\log L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i$$

STEP 3 Derivative $\log L(\lambda)$ with respect to λ equated to zero.

$$\frac{d}{d\lambda}\{\log L(\lambda)\} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Longrightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$$

STEP 4 Check that the second derivative of $\log L(\lambda)$ with respect to λ is negative at $\lambda = \hat{\lambda}$.

$$\frac{d^2}{d\lambda^2}\{\log L(\lambda)\} = -\frac{n}{\lambda^2} < 0 \quad \text{at} \quad \lambda = \hat{\lambda}$$

Now reparameterize in terms of $\tau = e^{-\lambda} \Longleftrightarrow \lambda = -\log \tau$.

STEP 1 Likelihood function $L(\tau)$, for $\tau > 0$,

$$L(\tau) = \prod_{i=1}^{n} f(x_i;\tau) = \prod_{i=1}^{n} [-\log \tau] e^{-[-\log \tau] x_i} = [-\log \tau]^n \exp\left\{-[-\log \tau] \sum_{i=1}^{n} x_i\right\}$$

STEP 2 Log-likelihood $\log L(\tau)$.

$$\log L(\tau) = n \log[-\log \tau] + \log \tau \sum_{i=1}^{n} x_i$$

STEP 3 Derivative $\log L(\tau)$ with respect to τ equated to zero.

$$\frac{d}{d\tau}\{\log L(\tau)\} = -\frac{n}{\tau \log \tau} + \frac{1}{\tau}\sum_{i=1}^{n} x_i = 0 \Longrightarrow \hat{\tau} = e^{-1/\bar{x}}.$$

STEP 4 Check that the second derivative of $\log L(\tau)$ with respect to τ is negative at $\tau = \hat{\tau}$.

Hence if $\tau = \tau(\lambda)$ then $\hat{\tau} = \tau(\hat{\lambda})$

2. Can use ML procedure to show that m.l.e.s of θ are

(a) $\dfrac{n}{-\sum_{i=1}^{n} \log x_i}$ (b) $\dfrac{n}{\sum_{i=1}^{n} \log x_i} - 1$ (c) $\dfrac{2n}{\sum_{i=1}^{n} x_i}$

3. Mles based on $x_1, ..., x_n$ obtained using standard ML techniques; can simplify likelihood by noting that

$$\sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2$$

which gives $\hat{\mu} = \bar{x}$ directly.

Mles based on $x_1, ..., x_n$ and $y_1, ..., y_m$ are obtained from the likelihood $L(\mu, \sigma_1, \sigma_2)$ where

$$L(\mu, \sigma_1, \sigma_2) = \prod_{i=1}^{n} f(x_i; \mu, \sigma_1) \prod_{i=1}^{m} f(y_i; \mu, \sigma_2)$$

so that

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i}{n+m} \qquad \hat{\sigma_1^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 \qquad \hat{\sigma_2^2} = \frac{1}{m}\sum_{i=1}^{m}(y_i - \hat{\mu})^2$$

4. Can show $\hat{\lambda} = \bar{x}$ (lecture notes p. 31); so the ML estimator is $\bar{X}$.

Now, if $X_1, ..., X_n \sim Poisson(\lambda)$ i.i.d., then can show that $Y = \sum_{i=1}^{n} X_i \sim Poisson(n\lambda)$, so the sampling distribution of the estimator is the distribution of random variable $T = \frac{1}{n}Y$, and hence

$$\mathrm{E}_{f_T}[T] = \frac{1}{n}\mathrm{E}_{f_Y}[Y] = \frac{1}{n}.n\lambda = \lambda \qquad \mathrm{Var}_{f_T}[T] = \frac{1}{n^2}\ \mathrm{Var}_{f_Y}[Y] = \frac{1}{n^2}.n\lambda = \frac{\lambda}{n}$$

as the expectation and variance of a $Poisson(n\lambda)$ random variable are both equal to $n\lambda$.

Now for $\epsilon > 0$,

$$\mathrm{P}[\lambda - \epsilon \leq T \leq \lambda + \epsilon] \ = \mathrm{P}\left[\ \lambda - \epsilon \ \leq \ \frac{1}{n}\sum_{i=1}^{n} X_i \ \leq \ \lambda + \epsilon\ \right] = \mathrm{P}\left[\ -\epsilon\sqrt{\frac{n}{\lambda}} \ \leq \ \frac{\sum_{i=1}^{n} X_i - n\lambda}{\sqrt{n\lambda}} \ \leq \ \epsilon\sqrt{\frac{n}{\lambda}}\ \right]$$

$$\approx \Phi\left(\epsilon\sqrt{\frac{n}{\lambda}}\right) - \left(-\epsilon\sqrt{\frac{n}{\lambda}}\right) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

by the Central Limit Theorem, so the probability distribution of the estimator T becomes more and more concentrated around λ as n gets larger and hence the estimator can be regarded as an good estimator of the parameter of interest.

5. Summary statistics : $\bar{x} = 6.4161$, $s^2 = (0.0742)^2$. Fix significance level at $\alpha = 0.05$.

(a) σ known; carry out a Z-test: Test statistic z given by

$$z = \frac{\bar{x} - c}{\sigma/\sqrt{n}} = \frac{6.4161 - 6.4}{0.03/\sqrt{15}} = 2.079 \sim N(0,1) \text{ if } H_0 \text{ is TRUE}$$

Critical Value : 1.96, so test statistic is more extreme than the critical value. Hence $\Longrightarrow$ REJECT H_0.

(b) σ unknown; carry out a T-test: Test statistic t given by

$$t = \frac{\bar{x} - c}{s/\sqrt{n}} = \frac{6.4161 - 6.4}{0.0742/\sqrt{15}} = 0.840 \sim t_{n-1} \text{ if } H_0 \text{ is TRUE}$$

where t_{n-1} is the Student-t distribution with $n - 1 = 14$ degrees of freedom.

Critical Value : 2.1448, so test statistic is not more extreme than the critical value. Hence $\Longrightarrow$ CANNOT REJECT H_0.

6. Summary statistics are

$$S_x = \sum_{i=1}^{n} x_i = 232.5 \quad S_y = \sum_{i=1}^{n} y_i = 229.0 \quad S_{xx} = \sum_{i=1}^{n} x_i^2 = 6974.25 \quad S_{xy} = \sum_{i=1}^{n} x_i y_i = 6884.65$$

(a) Estimates: $\hat{\alpha} = 0.9948$, $\hat{\beta} = -0.2281$.

(b) Fitted values: $\hat{y_i} = \hat{\alpha} + \hat{\beta} x_i$. Residuals: $e_i = y_i - \hat{y_i} = y_i - \hat{\alpha} - \hat{\beta} x_i$.

(c) $s^2 = \dfrac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{y_i})^2 = (0.2067)^2$

(d) Standard errors: $s.e.(\alpha) = 0.1378$, $s.e.(\beta) = 0.0052$.

(e) 95 % Confidence Intervals: $t_{n-2}(0.975) = 2.3060$

$$\begin{aligned} \text{Interval for } \alpha \quad &: \ (-0.6059 : 0.0297) \\ \text{Interval for } \beta \quad &: \ (0.9829 : 1.0068) \end{aligned}$$

95 % CI for α includes 0, so some evidence that $\alpha = 0$.

95 % CI for β includes 1, so some evidence that $\beta = 1$.

EXERCISES 5

1. Summary statistics : $\bar{x} = 276.89$, $s^2 = (27.08)^2$. Fix significance level at $\alpha = 0.05$.

(a) Test statistic q given by

$$q = \frac{(n-1)s^2}{\sigma^2} = \frac{18\ 733.4}{(30.0)^2} = 14.668 \sim \chi^2_{18} \text{ if } H_0 \text{ is TRUE}$$

Critical Values : $\chi^2_{18}(0.025) = 8.231$, $\chi^2_{18}(0.975) = 31.526$. Test statistic is not in critical region

$$\Longrightarrow \text{CANNOT REJECT } H_0.$$

(i) 95 % CI for μ : $\bar{x} \pm \dfrac{\sigma}{\sqrt{n}}\,1.96 = 276.89 \pm 13.49$.

(ii) 95 % CI for μ : $\bar{x} \pm \dfrac{s}{\sqrt{n}}\,t_{18}(0.975) = 276.89 \pm 13.05$.

95 % CI for σ: from above, if H_0 is TRUE

$$\mathrm{P}[8.231 \leq (n-1)s^2/\sigma^2 \leq 31.526] = 0.95 \Longleftrightarrow \mathrm{P}[(n-1)s^2/31.526 \leq \sigma^2 \leq (n-1)s^2/8.231] = 0.95$$

so a 95 % CI for σ^2 is given by

$$\mathrm{P}[418.74 \leq \sigma^2 \leq 1603.839] = 0.95 \Longleftrightarrow \mathrm{P}[20.46 \leq \sigma \leq 40.05] = 0.95.$$

2. $Q \sim F_{\nu_1,\nu_2}$. Then $\mathrm{P}[Q \leq x] = 0.975 \Longleftrightarrow \mathrm{P}[1/Q \geq 1/x] = 0.975 \Longleftrightarrow \mathrm{P}[1/Q < 1/x] = 0.025$

But $Q^* = 1/Q = \dfrac{V/\nu_2}{U/\nu_1} \sim F_{\nu_2,\nu_1}$. $\mathrm{P}[Q \leq x] = 0.975 \Longleftrightarrow \mathrm{P}[Q^* \leq 1/x] = 0.025$.

3. Summary statistics: $n_1 = 14$, $\bar{x_1} = 153.857$, $s_1^2 = (39.04)^2$, $n_2 = 5$, $\bar{x_2} = 113.000$, $s_2^2 = (56.08)^2$.

(i) Pooled estimate of variance, $s_P^2 = ((n_1-1)s_1^2 + (n_2-1)s_2^2)/(n_1+n_2-2) = (43.65)^2$.

Test statistic t given by

$$t = \frac{x_1 - x_2}{s_P\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{153.857 - 113.000}{43.65 \times 0.521} = 1.797 \sim t_{n_1+n_2-2} \text{ if } H_0 \text{ is TRUE}$$

Critical Value : $n_1 + n_2 - 2 = 17$, so $t_{17}(0.975) = 2.1098$. Test statistic is not in critical region

$$\Longrightarrow \text{CANNOT REJECT } H_0.$$

(ii) Test statistic q given by

$$q = \frac{s_1^2}{s_2^2} = \frac{(39.04)^2}{(56.08)^2} = 0.4846 \sim F_{n_1-1,n_2-1} \text{ if } H_0 \text{ is TRUE}$$

Critical Values : $F_{13,4}(0.025) = 1/(3.996) = 0.2502$, $F_{13,4}(0.975) = 8.715$. Test statistic is not in critical region

$$\Longrightarrow \text{CANNOT REJECT } H_0.$$

(iii) 95 % CI for $\mu_1 - \mu_2$:

$$(x_1 - x_2) \pm s_P\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\; t_{n_1+n_2-2}(0.975) = (153.857 - 113.000) \pm 43.65 \times 0.521 \times 2.1098$$

so the interval is (-7.123:88.837).

(iv) 95 % CI for σ_1/σ_2: from (ii) above, if H_0 is TRUE

$$\mathrm{P}\left[0.2502 \leq \frac{(s_1^2/\sigma_1^2)}{(s_2^2/\sigma_2^2)} \leq 8.755\right] = 0.95 \Longleftrightarrow \mathrm{P}\left[\frac{s_1^2}{s_2^2\ 8.755} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{s_1^2}{s_2^2\ 0.2502}\right] = 0.95$$

as test statistic q has an F-distribution. Thus a 95 % CI for σ_1/σ_2 is given by

$$\mathrm{P}\left[0.055 \leq \frac{\sigma_1^2}{\sigma_2^2} \leq 1.937\right] = 0.95 \Longleftrightarrow \mathrm{P}\left[0.235 \leq \frac{\sigma_1}{\sigma_2} \leq 1.392\right] = 0.95.$$

5. For probability plot, need to evaluate $q_1, ..q_n$, where

$$F_X(q_i) = \frac{i}{n+1} \Longleftrightarrow q_i^\alpha = -\frac{1}{\lambda}\log\left(1 - \frac{i}{n+1}\right)$$

For a probability plot, need to equate q_i to x_i (in sorted sample), that is, if the model is correct $x_i \approx q_i$, or equivalently

$$\log x_i \approx \log q_i = -\frac{1}{\alpha}\log\lambda + \frac{1}{\alpha}\log\left(-\log\left(1 - \frac{i}{n+1}\right)\right)$$

so that a plot of $\log x_i$ against $\log\left(-\log\left(1 - \frac{i}{n+1}\right)\right)$ should be a straight line with slope $1/\alpha$ and intercept $-\log\lambda/\alpha$.

6. Data mean is $242/200 = 1.21$. For each model need to calculate expected frequencies $E_i = 200p_i$ where p_i is defined by the model being tested, for $i = 1, ..., k$, where $k = 5$ categories. For (iii), use ML estimate of p, which is $\bar{x}/4 = 1.21/4 = 0.3025$; for (v), use ML estimate of λ, which is $\bar{x} = 1.21$. Relevant critical values of the test are only available at the $\alpha = 0.01$ significance level from the tables provided; $\chi_3^2(0.99) = 11.345$, $\chi_4^2(0.99) = 13.277$ (could use $\alpha = 0.05$ if you have the figures available).

For model (i), $p_i = 0.2$ for each category, degrees of freedom is $k - 0 - 1 = 4$.

For model (ii), p_i given by $Binomial(4, 0.25)$ formula, degrees of freedom is $k - 0 - 1 = 4$.

For model (iii), p_i given by $Binomial(4, 0.3025)$ formula, degrees of freedom is $k - 1 - 1 = 3$.

For model (iv), p_i given by $Poisson(1)$ formula, degrees of freedom is $k - 0 - 1 = 4$.

For model (v), p_i given by $Poisson(1.21)$ formula, degrees of freedom is $k - 1 - 1 = 3$.

Could proceed by pooling last two categories, reducing k to 4.