# CHEM. ENG. II - PROBABILITY AND STATISTICS

## EXERCISES 4

1. Suppose that $x_1, ..., x_n$ is a sample from an exponential distribution with parameter $\lambda$. Derive the m.l.e., $\hat{\lambda}$, of $\lambda$.

Now consider the function $\tau(\lambda)$ of $\lambda$, where $\tau$ is defined by

$$\tau(\lambda) = e^{-\lambda} = \mathrm{P}[X > 1]$$

where $X$ is the random variable corresponding to the outcome of a new but identical experiment. Write down the likelihood **in terms of** $\tau$, and thus find the m.l.e. of $\tau$.

2. Find the m.l.e.s of the parameter $\theta$ based on a sample $x_1, ..., x_n$ from each of the following probability models;

(a) $f(x;\theta) = \theta x^{\theta-1}$ $, 0 < x < 1$ for $\theta > 0$.

(b) $f(x;\theta) = (\theta+1)x^{-\theta-2}$ $, 1 < x$ for $\theta > 0$.

(c) $f(x;\theta) = \theta^2 x e^{-\theta x}$ $, 0 < x$ for $\theta > 0$.

3. Suppose that $x_1, ..., x_n$ and $y_1, ..., y_m$ are independent samples from Normal distributions with the same mean but different variances, that is,

$$\begin{array}{ll} x_1, ..., x_n & \text{are modelled using } N(\mu, \sigma_1^2) \\ y_1, ..., y_m & \text{are modelled using } N(\mu, \sigma_2^2). \end{array}$$

First, verify that the the m.l.e.s for $\mu$ and $\sigma_1^2$ **based on** $x_1, ..., x_n$ **only** are given by

$$\begin{aligned} \hat{\mu} &= \bar{x} \\ \hat{\sigma_1^2} &= S^2 \end{aligned}$$

respectively, and then find the m.l.e.s of $\mu$, $\sigma_1^2$ and $\sigma_2^2$ based on the **pooled** sample comprising both $x_1, ..., x_n$ and $y_1, ..., y_m$.

4. Find the maximum likelihood estimator, $\hat{\lambda}$, of parameter $\lambda$ derived from a random sample $X_1, ..., X_n$ from a $Poisson(\lambda)$ distribution. By considering the probability distribution of random variable

$$X = \sum_{i=1}^{n} X_i$$

find an approximation to

$$\mathrm{P}[\lambda - \epsilon \leq \hat{\lambda} \leq \lambda + \epsilon] \quad \text{for any } \epsilon > 0$$

for large $n$ using the Central Limit Theorem. What is the interpretation of this result ?

5. The pH of the catalyst in a chemical reaction is to be measured by a standard titration procedure. In order for the reaction to proceed at the desired rate, the catalyst pH must be very close to 6.4. The following measurements of pH were obtained;

| | | | | |
|---|---|---|---|---|
| 6.492 | 6.397 | 6.469 | 6.343 | 6.397 |
| 6.542 | 6.467 | 6.351 | 6.390 | 6.446 |
| 6.311 | 6.288 | 6.396 | 6.515 | 6.438 |

Assuming that the data are well-modelled by a normal distribution with mean $\mu$ and variance $\sigma^2$, carry out a test of the hypothesis

$$\begin{aligned} H_0 &: \mu = 6.4 \\ H_1 &: \mu \neq 6.4 \end{aligned}$$

at the 5 % significance level, if

(i) $\sigma$ is **known**, $\sigma = 0.03$,

(ii) $\sigma$ is **unknown**.

6. Determining small quantities of calcium in the presence of Magnesium is a difficult problem for analysts. Direct precipitation is not feasible, but one proposed procedure involves the use of alcohol as a solvent. The data in the first row below present the results of applying the alcohol method to ten mixtures containing known quantities of CaO ($x$, in mg). The second row gives, in each instance, the amount of CaO ($y$, in mg) recovered;

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 4.0 | 8.0 | 12.5 | 16.0 | 20.0 | 25.0 | 31.0 | 36.0 | 40.0 | 40.0 |
| $y$ | 3.7 | 7.8 | 12.1 | 15.6 | 19.8 | 24.5 | 31.1 | 35.5 | 39.4 | 39.5 |

Assuming a linear regression relationship between corresponding random variables $X$ and $Y$, that is, $\mathrm{E}[Y|X = x] = \alpha + \beta x$, and that the random errors are i.i.d normally distributed, calculate

(a) the m.l.e s of $\alpha$ and $\beta$,

(b) the fitted values and residuals from the resulting fit,

(c) the corrected estimate of variance, $s^2$

(d) the standard errors of the estimates of $\alpha$ and $\beta$,

(e) 95% confidence intervals for $\alpha$ and $\beta$.

On the basis of the confidence intervals, discuss whether or not there is evidence that $\alpha = 0$, or that $\beta = 1$.