CHEM. ENG. II - PROBABILITY AND STATISTICS

EXERCISES 3

1. A random variable X has a binomial distribution with $n = 10$ and $\theta = 0.2$. Calculate the probability that X takes each value in the sample space.

2. A fair coin is tossed 5 times. What is the probability of getting
   (i) 0 heads
   (ii) at least 4 heads
   (iii) an odd number of heads.

3. An industrial process manufactures machine components. If the components are produced independently, and the probability that a component is defective is $\theta = 0.01$, calculate the probability that a batch of 50 components contains at most two defectives. Now suppose that a different batch of 50 components is inspected, and labelled "1" to "50". Given that a total of three defectives are found, calculate the probability that the component labelled "1" is defective.

4. If X has a Poisson distribution with parameter λ, obtain a formula for $p_x = \mathrm{P}[X = x]$ in terms of p_{x-1}. Use the formula to compute the probabilities for $x = 0, 1, ...10$ if $\lambda = 5.0$.

5. Verify that the pdf for the exponential distribution satisfies the two conditions for a density function (i.e. that $f(x) \geq 0$ for $x \in \mathbb{X} = \mathbb{R}^+$, and $\int_{\mathbb{X}} f(x)\ dx = 1$). Carefully sketch the pdf when $\lambda = 0.1, 1, 10$.
If random variable X has an exponential distribution with rate parameter $\lambda = 2.0$, calculate the probability that
   (a) $X \leq 1$
   (b) $X > 4$

6. If $X_1, ..., X_n$ are independent random variables with $X_i \sim \mathit{Exponential}(\lambda)$, prove, using the Convolution Theorem, that the density function of random variable $Y_n = X_1 + ... + X_n$ is given by

$$f_{Y_n}(y) = \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y} \quad y > 0.$$

It is possible to calculate the probability $\mathrm{P}[Y_n \leq y]$ in a straightforward way ?

7. Prove that, if $X \sim N(\mu, \sigma^2)$, then

$$\mathrm{E}_{f_X}[X] \;=\; \mu, \qquad \mathrm{Var}_{f_X}[X] \;=\; \sigma^2$$

Prove also that, if $X \sim N(0, 1)$, and $Y = \sigma X + \mu$, then $Y \sim N(\mu, \sigma^2)$.
[Hint: consider $\mathrm{P}[\ Y \leq y\ ]$, which gives the cdf $F_Y(y)$, and then differentiate to find the pdf $f_Y(y)$.]

8. Use the statistical tables provided to calculate the following probabilities, first if X has a standard normal distribution then, if $X \sim N(0.5, 2.0^2)$.
   (i) $\mathrm{P}[\ X \leq 2.2\ ]$
   (ii) $\mathrm{P}[\ X > -0.75\ ]$
   (iii) $\mathrm{P}[\ -1.7 \leq X < -1\ ]$
   (iv) $\mathrm{P}[\ |X| > 1.96\ ]$

9. If $Y \sim \chi^2_1 = Gamma(1/2, 1/2)$, derive an expression for P[$Y > y$] in terms of the standard normal cdf, $\Phi(.)$.
Hint: If $Y \sim \chi^2_1$, then $X = \sqrt{Y} \sim N(0,1)$, so

$$\mathrm{P}[\ Y \leq y\ ] \ = \ \mathrm{P}[\ -\sqrt{y}\ \leq\ X\ \leq\ \sqrt{y}\ ]$$

10. Using a normal approximation based on the Central Limit Theorem, or otherwise, compute the following probabilities;

(i)	P[$X \leq 200$]	if $X \sim Binomial(1000, 0.2)$
(ii)	P[$X > 2275$]	if $X \sim Binomial(2500, 0.9)$
(iii)	P[$725 \leq X < 740$]	if $X \sim Binomial(1500, 0.5)$
(iv)	P[$X \leq 190$]	if $X \sim Poisson(200)$
(v)	P[$X > 120$]	if $X \sim Poisson(100)$
(vi)	P[$X \leq 12$]	if $X \sim Gamma(20, 2)$