CHEM. ENG. II - PROBABILITY AND STATISTICS

EXERCISES 2

1. For each of the following experimental situations, define a random variable X with corresponding range $\mathbb{X}$ that is suitable for describing the outcome of the experiment concerned;

(a) In testing, a mechanical component is installed at time $x = 0$, and it is recorded whether the component is still operating at time $x = 10$.

(b) Components are distributed from the production factory in batches of 100. Occasionally, components are found to be defective, that is, they do not function on initial testing. As part of a quality control exercise, each component in a particular batch is tested, and the number of defective components is recorded.

(c) A component is installed at time $x = 0$, and functions until failure at some subsequent time, when an identical component is installed, and so on. The number of components that fail in the first month of operation is recorded.

(d) The end-user of the components purchases a single batch of 100 components. The time from initial installation of the first component until the failure of the 100th component is recorded.

2. For a function f_X to be a valid probability mass function for a **discrete** random variable X that takes values on range $\mathbb{X} = \{x_1, x_2, ...\}$, it must satisfy the following two conditions;

(i) $f_X(x_i) \geq 0$ for all i,

(ii) $\sum_i f_X(x_i) = 1$.

State whether each of the following functions is a valid probability mass function for the given $\mathbb{X}$.

(a) $f_X(x) = (x+1)/15$, $x \in \{0, 1, 2, 3, 4\}$

(b) $f_X(x) = (1-x)/4$, $x \in \{-1, 0, 2\}$

(c) $f_X(x) = (1/4)^{x-1}(3/4)$, $x \in \{1, 2, 3, ...\}$

(d) $f_X(x) = (2x-1)/100$, $x \in \{1, 2, ..., 12\}$

3. For a function f_X to be a valid probability density function (p.d.f.) for a **continuous** random variable X that takes values on range $\mathbb{X}$, it must satisfy the following two conditions;

(i) $f_X(x) \geq 0$ for all $x \in \mathbb{X}$

(ii) $\int_{\mathbb{X}} f_X(x)\ dx = 1$.

State whether each of the following functions is a valid p.d.f. for the given $\mathbb{X}$.

(a) $f_X(x) = 3x^2$, $x \in \{x : 0 < x < 1\}$

(b) $f_X(x) = x^{-2}$, $x \in \{x : 1 < x < \infty\}$

(c) $f_X(x) = x^{-1}$, $x \in \{x : 1 < x < \infty\}$

(d) $f_X(x) = -\ln x$, $x \in \{x : 0 < x < 1\}$

4. A **discrete** random variable X takes non-negative integer values, and has cumulative distribution function (cdf), F_X, of the form

$$F_X(x) \;=\; 1 - (1/2)^{x+1} \qquad x \in \mathbb{X} = \{0, 1, 2, ...\}$$

with $F_X(x) = 0$ if $x < 0$, and $F_X(x + \delta) = F_X(x)$ for $0 \leq \delta < 1$ for $x \in \mathbb{X}$.

Sketch the cdf. Find and sketch the probability mass function, f_X, of X. Find also P[X is even]

5. A **continuous** random variable X has probability density function (p.d.f.) given by

$$f_X(x) \;=\; c(1-x)x^2 \qquad 0 < x < 1$$

Sketch this pdf. Find the value of c, and the form of the c.d.f., F_X corresponding to this p.d.f.. Verify that $F_X(0) = 0$ and $F_X(1) = 1$, and find P$[1/4 < X < 3/4]$.

6. A **continuous** random variable X has c.d.f. given by the following three functional forms. In each case use differentiation to find the form of the p.d.f.

(a)

$$F_X(x) \;=\; c(\alpha x^{\beta} - \beta x^{\alpha}) \qquad 0 \leq x \leq 1$$

for constant $1 \leq \beta < \alpha$ (Here, also compute the value of constant c).

(b)

$$F_X(x) \;=\; 1 - \exp\{-\alpha \;\tan x\} \qquad 0 \leq x \leq \pi/2$$

for constant $0 < \alpha < 1$.

(c)

$$F_X(x) \;=\; \frac{2\beta x}{(\beta^2 + x^2)} \qquad 0 \leq x \leq \beta$$

for some constant $\beta > 0$.

7. Consider an experiment where two dice are to be rolled. Let X_1 be the random variable representing the sum of the scores on the two dice, and let X_2 be the random variable representing the largest of the two scores. Write down the joint mass function of X_1 and X_2, the marginal mass functions of X_1 and X_2, and the conditional mass function of X_1 given $X_2 = 4$.

8. Suppose that X_1 and X_2 are continuous random variables with joint probability density function (pdf) given by

$$f_{X_1,X_2}(x_1, x_2) \;=\; 4x_1x_2 \qquad 0 < x_1, x_2 < 1.$$

Find the marginal p.d.f. of random variable X_1 by integrating out x_2 from the joint density function. Also find the marginal p.d.f. of X_2. Repeat the calculations if the joint p.d.f. is given instead by

$$f_{X_1,X_2}(x_1, x_2) \;=\; x_1 + x_2 \qquad 0 < x_1, x_2 < 1.$$