Question 4. Either prove or disprove each of the following statements. You may assume any results from the course that you like. By the way, the last part of this question is worth no marks so don't spend too much time on it during the test.

(a) If $a, b \in \mathbf{Z}$ and there exists $\lambda, \mu \in \mathbf{Z}$ such that $\lambda a + \mu b = 2$, then $\gcd(a, b) = 2$ (here gcd means greatest common divisor).

(b) 2^{42} is congruent to 2 modulo 43.

(c) If a, b, c are positive integers and $a \mid bc$ then either $a \mid b$ or $a \mid c$.

(d) 10001 and 2482 are coprime.

(e) All of the parts of this question are provably true.

(f) All the parts of this question are provably false.

Answer. Note to markers: of course you can ignore (f). For the others, apologies, because there will be several approaches which makes marking a bit harder. If anyone successfully factors one of the numbers in (d) then well done to them! Euclid is a much easier approach though.

(a) This is false – for example if $a = b = \lambda = \mu = 1$ then we have a counterexample.**(2 marks)**

(b) This is false – by Fermat's Little Theorem 2^{42} is congruent to 1 mod 43.**(2 marks)**

(c) This is false – if $a = 4$ and $b = c = 2$ then we have a counterexample.**(2 marks)**

(d) This is false. Indeed if we use Euclid's algorithm then we see $10001 - 4 \times 2482 = 10001 - 9928 = 73$, and a relatively painless long division shows that $2482/73$ is exactly 34.**(2 marks)**

(e) This is false. Indeed, all the parts have been false so far, so this can't be true (and even if someone has managed to get all the parts so far incorrect, it must still be false because parts (e) and (f) contradict each other so can't both be true).**(2 marks)**

(f) This part cannot be true, because then it would be false. So it must be false, although given that we have proved that all the other parts are false, this part must hence not be provably false. You might think that this argument is a proof that it's false, but actually it's only a meta-proof; it's in fact a proof that there is no proof that it's false. In particular it's impossible to prove either that it's true or false so it is impossible to do what is asked, which is a bit unfair but then again that's why it's worth no marks. Go and read about Goedel sentences if you want to know more. **(0 marks)**