Question 3. Let $S \subseteq \mathbf{R}$ be a set of real numbers.

(a) i) What does it mean to say that $x \in \mathbf{R}$ is an *upper bound* for S?

ii) What does it mean to say that $y \in \mathbf{R}$ is a *least upper bound* for S?

(b) Now say $T \subseteq S$ is a subset of S. Prove that if $x \in \mathbf{R}$ is an upper bound for S then x is also an upper bound for T.

(c) Let S be $\{x \in \mathbf{R} : x < 0\}$. Prove that 0 is a least upper bound for S. You may assume any standard facts about the real numbers in this part.

Answer.

Note to markers: I am expecting careful proofs, like I did in lectures. Mark strictly, because they should learn now what we are expecting, rather than learning in the final exam: if they are a bit vague, or not careful about in which elements are in which sets, or not careful about their logic and implication signs, then take a mark off and write something in red on their scripts. On the other hand if they just say something is obvious, and it actually *is* obvious (e.g., 0 is pretty obviously an upper bound for the set in (iii)), I would just assume that they are smart and not penalise them.

(a) i) $x \in \mathbf{R}$ is an *upper bound* for S if for all $s \in S$ we have $s \leq x$. **(1 mark)**

ii) $y \in \mathbf{R}$ is a *least upper bound* for S if two things hold: (1) y is an upper bound for S and (2) if $z \in \mathbf{R}$ is any other upper bound for S then $y \leq z$. **(2 marks)**

(b) Say x is an upper bound for S. We are asked to prove that x is also an upper bound for T, so we need to prove that if $t \in T$ then $t \leq x$. But $T \subseteq S$, so $t \in T$ implies $t \in S$, and x is an upper bound for S hence $t \leq x$, and because t was an arbitrary element of T we are done. **(2 marks)**

(c) First we prove that 0 is an upper bound for S. This is true because $s \in S$ implies $s < 0$ and hence $s \leq 0$. **(1 mark)**

Next we prove that 0 is the least upper bound for S. So say z is any other upper bound for S. We need to prove $0 \leq z$. We do this by contradiction. So assume $0 > z$. Consider $s = z/2$, the average of 0 and z. Then $z < z/2 < 0$ (standard facts) so $s \in S$ but $s > z$. This contradicts the fact that z is an upper bound for S, and the contradiction completes the proof. **(4 marks)**